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Complex Numbers and Quadratic Equations

IMAGINARY NUMBERS

If the square of a given number is negative then such a number is called an imaginary number.

For example, $\sqrt{-1}$, $\sqrt{-2}$, etc., are imaginary numbers.

We denote $\sqrt{-1}$ *by the Greek letter* iota 'ι', which is transliterated as 'i'.

Thus, $\sqrt{-4} = 2i$, $\sqrt{-9} = 3i$ and $\sqrt{-5} = i\sqrt{5}$, etc.

POWERS OF i We have

$$i^0 = 1, i^1 = i, i^2 = -1, i^3 = i^2 \times i = (-1) \times i = -i$$

and $I^4 = i^2 \times i^2 = (-1) \times (-1) = 1.$

Thus, we have

$$\boxed{i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1.}$$

Let us consider i^n, where n is a positive integer and $n > 4$.

On dividing n by 4, let the quotient be m and the remainder be r. Then,

$n = 4m + r$, where $0 \le r < 4$.

$\therefore \quad i^n = i^{4m+r} = i^{4m} \times i^r = (i^4)^m \times i^r = i^r. \quad [\because\ i^4 = 1]$

EXAMPLES (i) $i^{98} = i^{4\times 24+2} = (i^4)^{24} \times i^2 = i^2 = -1. \quad [\because\ i^4 = 1]$

(ii) $i^{-98} = \dfrac{1}{i^{98}} = \dfrac{1}{i^{98}} \times \dfrac{i^2}{i^2} = \dfrac{i^2}{i^{(98+2)}} = \dfrac{-1}{1} = -1. \quad [\because\ i^{100} = 1 \text{ and } i^2 = -1]$

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate:*

(i) i^{23} *(ii)* i^{998} *(iii)* i^{-998} *(iv)* i^{-71}

(v) $(\sqrt{-1})^{91}$ *(vi)* $(i^{37} \times i^{-61})$ *(vii)* i^{-1}

SOLUTION We have

(i) $i^{23} = i^{(4\times 5)+3} = (i^4)^5 \times i^3 = i^3 = -i. \quad [\because\ i^4 = 1]$

(ii) $i^{998} = i^{4\times 249+2} = (i^4)^{249} \times i^2 = (1 \times i^2) = i^2 = -1. \quad [\because\ i^4 = 1]$

(iii) $i^{-998} = \frac{1}{i^{998}} \times \frac{i^2}{i^2} = \frac{i^2}{i^{1000}} = \frac{-1}{1} = -1.$ $[\because\ i^{1000} = (i^4)^{250} = 1]$

(iv) $i^{-71} = \frac{1}{i^{71}} \times \frac{i}{i} = \frac{i}{i^{72}} = \frac{i}{(i^4)^{18}} = \frac{i}{1} = i.$ $[\because\ i^4 = 1]$

(v) $(\sqrt{-1})^{91} = i^{91} = i^{4\times 22+3} = (i^4)^{22} \times i^3 = 1 \times (-i) = -i.$ $[\because\ i^3 = i]$

(vi) $i^{37} = i^{4\times 9+1} = (i^4)^9 \times i = 1 \times i = i.$

$i^{-61} = \frac{1}{i^{61}} \times \frac{i^3}{i^3} = \frac{i^3}{i^{64}} = \frac{-i}{(i^4)^{16}} = \frac{-i}{1} = -i.$

$\therefore \quad (i^{37} + i^{-61}) = i + (-i) = 0.$

(vii) $i^{-1} = \frac{1}{i} = \frac{1}{i} \times \frac{i^3}{i^3} = \frac{-i}{i^4} = \frac{-i}{1} = -i.$

EXAMPLE 2 *Prove that:*

(i) $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0$

(ii) $i^{107} + i^{112} + i^{117} + i^{122} = 0$

(iii) $(1+i)^4 \times \left(1 + \frac{1}{i}\right)^4 = 16.$

SOLUTION We have

(i) $i^n + i^{n+1} + i^{n+2} + i^{n+3}$

$= i^n(1 + i + i^2 + i^3)$

$= i^n(1 + i - 1 - i) = (i^n \times 0) = 0.$ $[\because\ i^2 = -1 \text{ and } i^3 = -i]$

(ii) $i^{107} + i^{112} + i^{117} + i^{122}$

$= i^{107}(1 + i^5 + i^{10} + i^{15}) = i^{107}(1 + i^4 \times i + i^8 \times i^2 + i^{12} \times i^3)$

$= i^{107}(1 + i + i^2 + i^3)$ $[\because\ i^4 = 1, i^8 = 1 \text{ and } i^{12} = 1]$

$= i^{107}(1 + i - 1 - i) = (i^{107} \times 0) = 0.$ $[\because\ i^2 = -1, i^3 = -i]$

(iii) $(1+i)^4 \times \left(1 + \frac{1}{i}\right)^4$

$= (1+i)^4 \times \left(1 + \frac{1}{i} \times \frac{i}{i}\right)^4 = (1+i)^4(1-i)^4$ $[\because\ i^2 = -1]$

$= \{(1+i)(1-i)\}^4 = (1-i^2)^4 = \{1-(-1)\}^4 = 2^4 = 16.$

EXAMPLE 3 *Show that* $\left\{i^{23} + \left(\frac{1}{i}\right)^{29}\right\}^2 = -4.$

SOLUTION We have

$i^{23} = i^{(4\times 5+3)} = (i^4)^5 \times i^3 = 1 \times (-i) = -i.$ $[\because\ i^4 = 1 \text{ and } i^3 = -i]$

$\left(\frac{1}{i}\right)^{29} = \frac{1}{i^{29}} = \frac{1}{i^{29}} \times \frac{i^3}{i^3} = \frac{i^3}{i^{32}} = \frac{-i}{1} = -i.$ $[\because\ i^3 = -i \text{ and } i^{32} = 1]$

$\therefore \quad \left\{i^{23} + \left(\frac{1}{i}\right)^{29}\right\}^2 = (-i-i)^2 = (-2i)^2 = 4i^2 = 4 \times (-1) = -4.$

EXAMPLE 4 *Simplify:*

$$(i)\ (-2i)\left(\frac{1}{6}i\right) \qquad (ii)\ (-i)(3i)\left(\frac{-1}{6}i\right)^3 \qquad (iii)\ 4\sqrt{-4}+5\sqrt{-9}-3\sqrt{-16}$$

SOLUTION We have

$$\text{(i)}\ (-2i)\left(\frac{1}{6}i\right)=\left(-2\times\frac{1}{6}\right)\times i^2=\frac{-1}{3}\times(-1)=\frac{1}{3}.$$

$$\text{(ii)}\ (-i)(3i)\left(\frac{-1}{6}i\right)^3=(-3i^2)\left(\frac{-1}{216}i^3\right)$$

$$=(-3)\times(-1)\left[\frac{-1}{216}\times(-i)\right] \qquad [\because\ i^3=-i]$$

$$=\left(3\times\frac{1}{216}\times i\right)=\frac{1}{72}i.$$

$$\text{(iii)}\ 4\sqrt{-4}+5\sqrt{-9}-3\sqrt{-16}$$

$$=(4\times 2i)+(5\times 3i)-(3\times 4i) \quad [\because\ \sqrt{-4}=2i,\ \sqrt{-9}=3i,\ \sqrt{-16}=4i]$$

$$=(8i+15i-12i)=(11)i.$$

EXAMPLE 5 *Show that* $(-\sqrt{-1})^{4n+3}=i$, *where n is a positive integer.*

SOLUTION We have

$$(-\sqrt{-1})^{4n+3}=(-i)^{4n+3} \qquad [\because\ \sqrt{-1}=i]$$

$$=(-i)^{4n}\times(-i)^3$$

$$=\{(-i)^4\}^n\times(-i^3)=(1\times i)=i.$$

$$[\because\ (-i)^4=1 \text{ and } -i^3=-(-i)=i]$$

Hence, $(-\sqrt{-1})^{4n+3}=i$.

EXAMPLE 6 *Show that the sum* $(1+i^2+i^4+\ldots+i^{2n})$ *is* 0 *when n is odd and* 1 *when n is even.*

SOLUTION Let $S=1+i^2+i^4+\ldots+i^{2n}$.

This is clearly a GP having $(n+1)$ terms with $a=1$ and $r=i^2=-1$.

$$\therefore\quad S=\frac{a(1-r^{n+1})}{(1-r)}=\frac{1\times\{1-(i^2)^{n+1}\}}{(1-i^2)}$$

$$=\frac{\{1-(-1)^{n+1}\}}{1-(-1)}=\frac{\{1-(-1)^{n+1}\}}{2}$$

$$=\begin{cases}\frac{1}{2}(1-1)=0, \text{ when } n \text{ is odd}\\ \frac{1}{2}(1+1)=1, \text{ when } n \text{ is even.}\end{cases}$$

AN IMPORTANT RESULT

For any two real numbers *a* and *b*, the result $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$ is true only when at least one of the given numbers is either zero or positive.

Thus, $\sqrt{-2}\times\sqrt{-3}=\sqrt{(-2)\times(-3)}=\sqrt{6}$ is wrong.

In fact, $\sqrt{-2}\times\sqrt{-3}=(i\sqrt{2})(i\sqrt{3})=i^2\times\sqrt{6}=-\sqrt{6}$.

EXAMPLE 7 *Explain the fallacy:*

$$-1 = (i \times i) = \sqrt{-1} \times \sqrt{-1} = \sqrt{(-1)\times(-1)} = \sqrt{1} = 1.$$

SOLUTION We know that for any real numbers a and b, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is true only when at least one of a and b is either 0 or positive.

$\therefore \quad \sqrt{-1} \times \sqrt{-1} \neq \sqrt{(-1)\times(-1)}$.

EXAMPLE 8 *Evaluate:*

(i) $\sqrt{-25} \times \sqrt{-49}$ *(ii)* $\sqrt{-36} \times \sqrt{16}$

SOLUTION We have

(i) $\sqrt{-25} \times \sqrt{-49} = (5i) \times (7i) = (35 \times i^2) = 35 \times (-1) = -35.$

(ii) $\sqrt{-36} \times \sqrt{16} = (6i) \times 4 = 24i.$

EXAMPLE 9 *Evaluate* $\sqrt{-16} + 3\sqrt{-25} + \sqrt{-36} - \sqrt{-625}$.

SOLUTION We have

$$\sqrt{-16} + 3\sqrt{-25} + \sqrt{-36} - \sqrt{-625}$$
$$= (4i + 3 \times 5i + 6i - 25i) = (4i + 15i + 6i - 25i) = 0.$$

EXERCISE 5A

1. Evaluate: (i) i^{19} (ii) i^{62} (iii) i^{373}

2. Evaluate: (i) $(\sqrt{-1})^{192}$ (ii) $(\sqrt{-1})^{93}$ (iii) $(\sqrt{-1})^{30}$

3. Evaluate: (i) i^{-50} (ii) i^{-9} (iii) i^{-131}

4. Evaluate: (i) $\left(i^{41} + \frac{1}{i^{71}}\right)$ (ii) $\left(i^{53} + \frac{1}{i^{53}}\right)$

Prove that:

5. $1 + i^2 + i^4 + i^6 = 0.$

6. $6i^{50} + 5i^{33} - 2i^{15} + 6i^{48} = 7i.$

7. $\frac{1}{i} - \frac{1}{i^2} + \frac{1}{i^3} - \frac{1}{i^4} = 0.$

8. $(1 + i^{10} + i^{20} + i^{30})$ is a real number.

9. $\left\{i^{21} - \left(\frac{1}{i}\right)^{46}\right\}^2 = 2i.$

10. $\left\{i^{18} + \left(\frac{1}{i}\right)^{25}\right\}^3 = 2(1 - i).$

11. $(1-i)^n\left(1 - \frac{1}{i}\right)^n = 2^n$ for all values of $n \in N$.

12. $\sqrt{-16} + 3\sqrt{-25} + \sqrt{-36} - \sqrt{-625} = 0.$

13. $(1 + i^2 + i^4 + i^6 + i^8 + \ldots + i^{20}) = 1.$

14. $i^{53} + i^{72} + i^{93} + i^{102} = 2i.$

15. $\sum_{n=1}^{13} (i^n + i^{n+1}) = (-1 + i),\ n \in N.$

ANSWERS (EXERCISE 5A)

1. (i) $-i$ (ii) -1 (iii) i

2. (i) 1 (ii) i (iii) -1

3. (i) -1 (ii) $-i$ (iii) i

4. (i) $2i$ (ii) 0

7. $\frac{1}{i}=\frac{1}{i}\times\frac{i^3}{i^3}=i^3=-i,\ \frac{1}{i^2}=\frac{1}{-1}=-1,\ \frac{1}{i^3}=\frac{1}{i^3}\times\frac{i}{i}=i$ and $\frac{1}{i^4}=1.$

$\therefore\quad \left\{\frac{1}{i}-\frac{1}{i^2}+\frac{1}{i^3}-\frac{1}{i^4}\right\}=(-i+1+i-1)=0.$

8. $(1+i^{10}+i^{20}+i^{30})=\{1+i^{4\times2+2}+i^{4\times5}+i^{4\times7+2}\}$

$=\{1+(i^4)^2\times i^2+(i^4)^5+(i^4)^7\times i^2\}$

$=(1+i^2+1+i^2)=(1-1+1-1)=0.$

11. $\frac{1}{i}=\frac{1}{i}\times\frac{i^3}{i^3}=\frac{i^3}{i^4}=i^3=-i.$

$\therefore\quad (1-i)^n\left(1-\frac{1}{i}\right)^n=(1-i)^n(1+i)^n \qquad \left[\because\ \frac{1}{i}=-i\right]$

$=\{(1-i)(1+i)\}^n=(1-i^2)^n=2^n. \qquad [\because\ i^2=-1]$

13. This is a GP in which $a=1, r=i^2=-1$ and $n=11.$

$\therefore\quad S=\frac{a(1-r^n)}{(1-r)}=\frac{1\times\{1-(-1)^{11}\}}{\{1-(-1)\}}=\frac{2}{2}=1.$

14. $i^{53}+i^{72}+i^{93}+i^{102}=(i^4)^{13}\times i+(i^4)^{18}+(i^4)^{23}\times i+(i^4)^{25}\times i^2$

$=(1\times i)+1+(1\times i)+1\times(-1)=i+1+i-1=2i.$

15. $\sum_{n=1}^{13}(i^n+i^{n+1})=\sum_{n=1}^{13}i^n(1+i)=(1+i)\cdot\sum_{n=1}^{13}i^n$

$=(1+i)\cdot\{i+i^2+i^3+\ldots+i^{13}\}$

$=(1+i)\frac{a(1-r^n)}{(1-r)}$, where $a=i, r=i$ and $n=13$

$=(1+i)\cdot\frac{i(1-i^{13})}{(1-i)}=i(1+i)\cdot\frac{(1-i)}{(1-i)}=(-1+i).\ [\because\ i^{13}=(i^4)^3\times i=1\times i=i]$

COMPLEX NUMBERS

COMPLEX NUMBERS *The numbers of the form* $(a+ib)$, *where* a *and* b *are real numbers and* $i=\sqrt{-1}$, *are known as complex numbers.* The set of all complex numbers is denoted by C.

$\therefore\quad C=\{(a+ib):a,b\in R\}.$

EXAMPLES Each of the numbers $(5+8i), (-3+\sqrt{2}\,i)$ and $\left(\frac{2}{3}-\frac{5}{7}i\right)$ is a complex number.

For a complex number, $z=(a+ib)$, *we have*

a = *real part of* z, written as $\mathrm{Re}(z)$

and b = *imaginary part of* z, written as $\mathrm{Im}(z)$.

EXAMPLES (i) If $z = (5 + 9i)$ then $\text{Re}(z) = 5$ and $\text{Im}(z) = 9$.

(ii) If $z = \left(\frac{2}{3} - 3i\right)$ then $\text{Re}(z) = \frac{2}{3}$ and $\text{Im}(z) = -3$.

(iii) If $z = \left(-7 + \frac{5}{8}i\right)$ then $\text{Re}(z) = -7$ and $\text{Im}(z) = \frac{5}{8}$.

PURELY REAL AND PURELY IMAGINARY NUMBERS

A complex number z is said to be

(i) *purely real*, if $\text{Im}(z) = 0$,

(ii) *purely imaginary*, if $\text{Re}(z) = 0$.

Thus, each of the numbers $2, -7, \sqrt{3}$ is *purely real*.

And, each of the numbers $2i, (\sqrt{3}\,i), \left(\frac{-3}{2}i\right)$ is *purely imaginary*.

CONJUGATE OF A COMPLEX NUMBER

Conjugate of a complex number $z = (a + ib)$ *is defined as,* $\bar{z} = (a - ib)$.

EXAMPLES (i) $\overline{(3 + 8i)} = (3 - 8i)$ (ii) $\overline{(-6 - 2i)} = (-6 + 2i)$ (iii) $-\overline{3} = -3$.

MODULUS OF A COMPLEX NUMBER

Modulus of a complex number $z = (a + ib)$, *denoted by* $|z|$, *is defined as* $|z| = \sqrt{a^2 + b^2}$.

EXAMPLES (i) If $z = (2 + 3i)$ then $|z| = \sqrt{2^2 + 3^2} = \sqrt{13}$.

(ii) If $z = (-5 - 4i)$ then $|z| = \sqrt{(-5)^2 + (-4)^2} = \sqrt{41}$.

EQUALITY OF COMPLEX NUMBERS If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ then

$$z_1 = z_2 \Leftrightarrow a_1 = a_2 \text{ and } b_1 = b_2.$$

EXAMPLE *If* $2y + (3x - y)i = (5 - 2i)$, *find the values of x and y.*

SOLUTION Equating the real and imaginary parts, we get

$$2y + (3x - y)i = (5 - 2i)$$

$$\Leftrightarrow \quad 2y = 5 \quad \text{and} \quad 3x - y = -2$$

$$\Leftrightarrow \quad y = \frac{5}{2} \quad \text{and} \quad 3x - \frac{5}{2} = -2$$

$$\Leftrightarrow \quad y = \frac{5}{2} \quad \text{and} \quad x = \frac{1}{6}.$$

Hence, $x = \frac{1}{6}$ and $y = \frac{5}{2}$.

SUM AND DIFFERENCE OF COMPLEX NUMBERS

If $z_1 = (a_1 + ib_1)$ *and* $z_2 = (a_2 + ib_2)$ *then we define:*

(i) $z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$

(ii) $z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2)$.

EXAMPLES (i) If $z_1 = (3+5i)$ and $z_2 = (-5+2i)$ then

$$z_1 + z_2 = \{3+(-5)\} + i\{5+2\} = (-2+7i)$$

and $z_1 - z_2 = \{3-(-5)\} + i(5-2) = (8+3i)$.

(ii) If $z_1 = (-6-2i)$ and $z_2 = (-3-5i)$ then

$$z_1 + z_2 = \{(-6)+(-3)\} + i\{(-2)+(-5)\} = (-9-7i)$$

and $z_1 - z_2 = \{-6-(-3)\} + i\{(-2)-(-5)\} = (-3+3i)$.

REMARKS (i) $\text{Re}(z_1+z_2) = \text{Re}(z_1) + \text{Re}(z_2)$ and $\text{Im}(z_1+z_2) = \text{Im}(z_1) + \text{Im}(z_2)$.

(ii) $\text{Re}(z_1-z_2) = \text{Re}(z_1) - \text{Re}(z_2)$ and $\text{Im}(z_1-z_2) = \text{Im}(z_1) - \text{Im}(z_2)$.

PROPERTIES OF ADDITION OF COMPLEX NUMBERS

(i) CLOSURE PROPERTY

The sum of two complex numbers is always a complex number.

Let $z_1 = (a_1 + ib_1)$ and $z_2 = (a_2 + ib_2)$ be any two complex numbers.

Then, $z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2)$

$= (a_1 + a_2) + i(b_1 + b_2)$, which is a complex number.

Thus, if z_1 and z_2 are any two complex numbers then $(z_1 + z_2)$ is also a complex number.

(ii) COMMUTATIVE LAW

For any two complex numbers z_1 and z_2, prove that

$$z_1 + z_2 = z_2 + z_1.$$

PROOF Let $z_1 = (a_1 + ib_1)$ and $z_2 = (a_2 + ib_2)$, where a_1, a_2, b_1, b_2 are real numbers.

$$\therefore \quad z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2)$$
$$= (a_1 + a_2) + i(b_1 + b_2)$$
$$= (a_2 + a_1) + i(b_2 + b_1) \quad \text{[by commutative law of addition in } R\text{]}$$
$$= (a_2 + ib_2) + (a_1 + ib_1) = z_2 + z_1.$$

Thus, $z_1 + z_2 = z_2 + z_1$ for all $z_1, z_2 \in C$.

Hence, addition of complex numbers is commutative.

(iii) ASSOCIATIVE LAW

For any complex numbers z_1, z_2 and z_3, prove that

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3).$$

PROOF Let $z_1 = (a_1 + ib_1)$, $z_2 = (a_2 + ib_2)$ and $z_3 = (a_3 + ib_3)$, where a_1, a_2, a_3 and b_1, b_2, b_3 are all real numbers.

$$\therefore \quad (z_1 + z_2) + z_3 = \{(a_1 + ib_1) + (a_2 + ib_2)\} + (a_3 + ib_3)$$
$$= \{(a_1 + a_2) + i(b_1 + b_2)\} + (a_3 + ib_3)$$
$$= \{(a_1 + a_2) + a_3\} + i\{(b_1 + b_2) + b_3\}$$
$$= \{a_1 + (a_2 + a_3)\} + i\{b_1 + (b_2 + b_3\}$$
$$\text{[by associative law of addition in } R\text{]}$$
$$= (a_1 + ib_1) + \{(a_2 + a_3) + i(b_2 + b_3)\}$$

$$= (a_1 + ib_1) + \{(a_2 + ib_2) + (a_3 + ib_3)\}$$
$$= z_1 + (z_2 + z_3).$$

$\therefore \quad (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ for all $z_1, z_2, z_3 \in C$.

Hence, addition of complex numbers is associative.

(iv) EXISTENCE OF ADDITIVE IDENTITY

For any complex number z, prove that

$$z + 0 = 0 + z = z.$$

PROOF Let $z = (a + ib)$ and we may write, $0 = (0 + i0)$. Then,

$$z + 0 = (a + ib) + (0 + i0) = (a + 0) + i(b + 0) = (a + ib)$$

and $0 + z = (0 + i0) + (a + ib) = (0 + a) + i(0 + b) = (a + ib)$.

$\therefore \quad z + 0 = 0 + z = z$ for all values of $x \in C$.

Thus, 0 is the additive identity for complex numbers.

(v) EXISTENCE OF ADDITIVE INVERSE

For every complex number z, prove that

$$z + (-z) = (-z) + z = 0.$$

PROOF Let $z = (a + ib)$. Then, $(-z) = (-a) + i(-b)$.

$\therefore \quad z + (-z) = (a + ib) + \{(-a) + i(-b)\}$

$$= \{a + (-a)\} + i\{b + (-b)\} = (0 + i0).$$

Similarly, $(-z) + z = (0 + i0) = 0$.

Hence, $z + (-z) = (-z) + z = 0$.

Thus, every complex number z has $(-z)$ as its additive inverse.

MULTIPLICATION OF COMPLEX NUMBERS

Let $z_1 = (a_1 + ib_1)$ and $z_2 = (a_2 + ib_2)$. Then, we define

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2).$$

$\therefore \quad z_1 z_2 = \{\mathrm{Re}(z_1) \cdot \mathrm{Re}(z_2) - \mathrm{Im}(z_1) \cdot \mathrm{Im}(z_2)\}$

$$+ i\{\mathrm{Re}(z_1) \cdot \mathrm{Im}(z_2) + \mathrm{Re}(z_2) \cdot \mathrm{Im}(z_1)\}.$$

EXAMPLES (i) Let $z_1 = (3 + 2i)$ and $z_2 = (5 + 4i)$. Then,

$$z_1 z_2 = \{(3 \times 5) - (2 \times 4)\} + i\{(3 \times 4) + (2 \times 5)\}$$
$$= (15 - 8) + i(12 + 10) = (7 + 22i).$$

(ii) Let $z_1 = (-2 + 3i)$ and $z_2 = (7 - 5i)$. Then,

$$z_1 z_2 = \{(-2) \times 7 - 3 \times (-5)\} + i\{(-2) \times (-5) + 3 \times 7\}$$
$$= (-14 + 15) + i(10 + 21) = (1 + 31i).$$

PROPERTIES OF MULTIPLICATION OF COMPLEX NUMBERS

(i) CLOSURE PROPERTY *The product of two complex numbers is always a complex number.*

PROOF Let $z_1 = (a_1 + ib_1)$ and $z_2 = (a_2 + ib_2)$ be any two complex numbers.

Then, a_1, a_2 and b_1, b_2 are real numbers.

$\therefore \quad z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2)$

$= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)$, which is a complex number.

Thus, the product of two complex numbers is always a complex number.

(ii) COMMUTATIVE LAW

For any two complex numbers z_1 and z_2, *prove that*

$$z_1 z_2 = z_2 z_1.$$

PROOF Let $z_1 = (a_1 + ib_1)$ and $z_2 = (a_2 + ib_2)$. Then,

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)$$

$$= (a_2 a_1 - b_2 b_1) + i(b_2 a_1 + a_2 b_1)$$

[by commutativity of multiplication on *R*]

$$= (a_2 + ib_2)(a_1 + ib_1) = z_2 z_1.$$

$\therefore \quad z_1 z_2 = z_2 z_1$ for all $z_1, z_2 \in C$.

Hence, multiplication of complex numbers is commutative.

(iii) ASSOCIATIVE LAW

For any three complex numbers z_1, z_2 *and* z_3, *prove that*

$$(z_1 z_2) z_3 = z_1 (z_2 z_3).$$

PROOF Let $z_1 = (a_1 + ib_1)$, $z_2 = (a_2 + ib_2)$ and $z_3 = (a_3 + ib_3)$ be any three complex numbers. Then a_1, a_2, a_3 and b_1, b_2, b_3 are all real numbers.

$\therefore \quad (z_1 z_2) z_3 = \{(a_1 + ib_1)(a_2 + ib_2)\}(a_3 + ib_3)$

$$= \{(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)\}(a_3 + ib_3)$$

$$= \{(a_1 a_2 - b_1 b_2) a_3 - (a_1 b_2 + b_1 a_2) b_3\}$$

$$+ i\{(a_1 a_2 - b_1 b_2) b_3 + (a_1 b_2 + b_1 a_2) a_3\}$$

$$= \{a_1 (a_2 a_3 - b_2 b_3) - b_1 (a_2 b_3 + b_2 a_3)\}$$

$$+ i\{a_1 (a_2 b_3 + a_3 b_2) + b_1 (a_2 a_3 - b_2 b_3)\}$$

$$= (a_1 + ib_1)\{(a_2 a_3 - b_2 b_3) + i(a_2 b_3 + a_3 b_2)\}$$

$$= (a_1 + ib_1)\{(a_2 + ib_2)(a_3 + ib_3)\} = z_1 (z_2 z_3).$$

Thus, $(z_1 z_2) z_3 = z_1 (z_2 z_3)$ for all $z_1, z_2, z_3 \in C$.

Hence, multiplication of complex numbers is associative.

(iv) EXISTENCE OF MULTIPLICATIVE IDENTITY

The complex number $(1 + i0)$ is the multiplicative identity in *C*.

Let $z = (a + ib)$. Then,

$$z \times 1 = (a + ib)(1 + i0) = \{(a \times 1) - (b \times 0)\} + i\{(a \times 0) + (b \times 1)\} = (a + ib) = z.$$

Similarly, $1 \times z = z$.

Thus, $(z \times 1) = (1 \times z) = z$ for all $z \in C$.

Hence, the complex number $1 = (1 + i0)$ is the multiplicative identity.

(v) EXISTENCE OF MULTIPLICATIVE INVERSE

Let $z = (a + ib)$. Then,

$$z^{-1} = \frac{1}{z} = \frac{1}{(a+ib)} = \frac{1}{(a+ib)} \times \frac{(a-ib)}{(a-ib)} = \frac{(a-ib)}{(a^2+b^2)}.$$

Clearly, $z \times z^{-1} = z^{-1} \times z = 1$.

Thus, every $z = (a + ib)$ has its multiplicative inverse, given by

$$z^{-1} = \frac{1}{z} = \frac{(a-ib)}{(a^2+b^2)} = \frac{\bar{z}}{|z|^2}. \qquad \left[\text{Remember: } z^{-1} = \frac{\bar{z}}{|z|^2}\right]$$

$\therefore \quad z\bar{z} = |z|^2.$

THINGS TO REMEMBER

(i) $z = (a + ib) \Rightarrow \bar{z} = (a - ib)$ and $|z|^2 = (a^2 + b^2)$.

(ii) $z = (a + ib) \Rightarrow z^{-1} = \dfrac{z}{|z|^2} = \dfrac{(a-ib)}{(a^2+b^2)}$.

(vi) DISTRIBUTIVE LAWS

For complex numbers z_1, z_2, z_3, prove that

$$z_1 \cdot (z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

$$(z_1 + z_2) \cdot z_3 = z_1 z_3 + z_2 z_3.$$

PROOF Let $z_1 = (a_1 + ib_1)$, $z_2 = (a_2 + ib_2)$ and $z_3 = (a_3 + ib_3)$ be any three complex numbers. Then, a_1, a_2, a_3 and b_1, b_2, b_3 are real numbers.

$$\begin{aligned}
\therefore \quad & z_1 \cdot (z_2 + z_3) \\
&= (a_1 + ib_1)[(a_2 + ib_2) + (a_3 + ib_3)] \\
&= (a_1 + ib_1)[(a_2 + a_3) + i(b_2 + b_3)] \\
&= \{a_1(a_2 + a_3) - b_1(b_2 + b_3)\} + i\{a_1(b_2 + b_3) + b_1(a_2 + a_3)\} \\
&= \{(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)\} + \{(a_1 a_3 - b_1 b_3) + i(a_1 b_3 + b_1 a_3)\} \\
&= z_1 z_2 + z_1 z_3.
\end{aligned}$$

Thus, $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ for all $z_1, z_2, z_3 \in C$.

Similarly, we can prove that

$$(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3.$$

DIVISION OF TWO COMPLEX NUMBERS

Let z_1 and z_2 be complex numbers such that $z_2 \neq 0$.

Then, $\dfrac{z_1}{z_2} = z_1 \cdot \dfrac{1}{z_2} = z_1 z_2^{-1}$.

EXAMPLE *Find $\dfrac{z_1}{z_2}$, when $z_1 = (6 + 3i)$ and $z_2 = (3 - i)$.*

SOLUTION We have $\dfrac{z_1}{z_2} = z_1 z_2^{-1}$.

$$\text{Now, } z_2^{-1} = \frac{\bar{z}_2}{|z_2|^2} = \frac{\overline{(3-i)}}{|3-i|^2} = \frac{(3+i)}{\{3^2+(-1)^2\}} = \frac{(3+i)}{10}.$$

$$\therefore \quad \frac{z_1}{z_2} = z_1 \cdot z_2^{-1} = (6+3i)\cdot\frac{(3+i)}{10} = \frac{(6+3i)(3+i)}{10}$$

$$= \frac{15+15i}{10} = \frac{15(1+i)}{10} = \frac{3(1+i)}{2}.$$

SOME IDENTITIES ON COMPLEX NUMBERS

THEOREM 1 *For any complex numbers z_1 and z_2, prove that:*

(i) $(z_1+z_2)^2 = z_1^2+z_2^2+2z_1z_2$

(ii) $(z_1-z_2)^2 = z_1^2+z_2^2-2z_1z_2$

(iii) $(z_1^2-z_2^2) = (z_1+z_2)(z_1-z_2)$

(iv) $(z_1+z_2)^3 = z_1^3+z_2^3+3z_1z_2(z_1+z_2)$

(v) $(z_1-z_2)^3 = z_1^3-z_2^3-3z_1z_2(z_1-z_2)$

PROOF We have

(i) $(z_1+z_2)^2 = (z_1+z_2)(z_1+z_2)$

$= (z_1+z_2)z_1+(z_1+z_2)z_2$ [by distributive law]

$= z_1^2+z_2z_1+z_1z_2+z_2^2$ [by distributive law]

$= z_1^2+2z_1z_2+z_2^2$ [$\because\ z_2z_1 = z_1z_2$]

$= z_1^2+z_2^2+2z_1z_2.$

$\therefore \quad (z_1+z_2)^2 = z_1^2+z_2^2+2z_1z_2.$

(ii) $(z_1-z_2)^2 = (z_1-z_2)(z_1-z_2)$

$= (z_1-z_2)z_1-(z_1-z_2)z_2$ [by distributive law]

$= z_1^2-z_2z_1-z_1z_2+z_2^2$ [by distributive law]

$= z_1^2-2z_1z_2+z_2^2$ [$\because\ z_2z_1 = z_1z_2$]

$= z_1^2+z_2^2-2z_1z_2.$

$\therefore \quad (z_1-z_2)^2 = z_1^2+z_2^2-2z_1z_2.$

(iii) RHS $= (z_1+z_2)(z_1-z_2)$

$= (z_1+z_2)z_1-(z_1+z_2)z_2$ [by distributive law]

$= z_1^2+z_2z_1-z_1z_2-z_2^2$ [by distributive law]

$= (z_1^2-z_2^2) = \text{LHS}.$ [$\because\ z_2z_1 = z_1z_2$]

$\therefore \quad (z_1^2-z_2^2) = (z_1+z_2)(z_1-z_2).$

(iv) $(z_1+z_2)^3=(z_1+z_2)^2\cdot(z_1+z_2)$

$=(z_1^2+z_2^2+2z_1z_2)(z_1+z_2)$ [expanding $(z_1+z_2)^2$]

$=(z_1^2+z_2^2+2z_1z_2)z_1+(z_1^2+z_2^2+2z_1z_2)z_2$

[by distributive law]

$=z_1^3+z_2^2z_1+2z_1^2z_2+z_1^2z_2+z_2^3+2z_1z_2^2$

$=z_1^3+z_2^3+3z_1^2z_2+3z_1z_2^2=z_1^3+z_2^3+3z_1z_2(z_1+z_2).$

$\therefore\quad (z_1+z_2)^3=z_1^3+z_2^3+3z_1z_2(z_1+z_2).$

(v) LHS $=(z_1-z_2)^3$

$=(z_1-z_2)^2(z_1-z_2)=(z_1^2-2z_1z_2+z_2^2)(z_1-z_2)$

$=(z_1^2-2z_1z_2+z_2^2)z_1-(z_1^2-2z_1z_2+z_2^2)z_2$

$=z_1^3-2z_1^2z_2+z_2^2z_1-z_1^2z_2+2z_1z_2^2-z_2^3$

$=z_1^3-3z_1^2z_2+3z_1z_2^2-z_2^3=\text{RHS}.$

This may be written as $(z_1-z_2)^3=z_1^3-z_2^3-3z_1z_2(z_1-z_2).$

SOLVED EXAMPLES

EXAMPLE 1 *Simplify:*

(i) $3(6+6i)+i(6+6i)$ *(ii)* $(1-i)-(-3+6i)$

(iii) $\left(\frac{1}{3}-\frac{2}{3}i\right)-\left(4+\frac{3}{2}i\right)$ *(iv)* $\left\{\left(\frac{1}{5}+\frac{7}{5}i\right)-\left(6+\frac{1}{5}i\right)\right\}-\left(\frac{-4}{5}+i\right)$

SOLUTION We have

(i) $3(6+6i)+i(6+6i)=18+18i+6i+6i^2=18+24i-6=12+24i.$

(ii) $(1-i)-(-3+6i)=1-i+3-6i=4-7i.$

(iii) $\left(\frac{1}{3}-\frac{2}{3}i\right)-\left(4+\frac{3}{2}i\right)=\frac{1}{3}-\frac{2}{3}i-4-\frac{3}{2}i$

$=\left(\frac{1}{3}-4\right)-\left(\frac{2}{3}+\frac{3}{2}\right)i=\frac{-11}{3}-\frac{13}{6}i.$

(iv) $\left\{\left(\frac{1}{5}+\frac{7}{5}i\right)-\left(6+\frac{1}{5}i\right)\right\}-\left(\frac{-4}{5}+i\right)$

$=\left(\frac{1}{5}+\frac{7}{5}i-6-\frac{1}{5}i\right)-\left(\frac{-4}{5}+i\right)=\left\{\left(\frac{1}{5}-6\right)+\left(\frac{7}{5}-\frac{1}{5}\right)i\right\}-\left(\frac{-4}{5}+i\right)$

$=\frac{-29}{5}+\frac{6}{5}i+\frac{4}{5}-i=\left(\frac{-29}{5}+\frac{4}{5}\right)+\left(\frac{6}{5}i-i\right)=\left(-5+\frac{1}{5}i\right).$

EXAMPLE 2 *Express* $(-\sqrt{3}+\sqrt{-2})(2\sqrt{3}-i)$ *in the form* $(a+ib)$.

SOLUTION We have

$(-\sqrt{3}+\sqrt{-2})(2\sqrt{3}-i)=(-\sqrt{3}+i\sqrt{2})(2\sqrt{3}-i)$

$=-6+i\sqrt{3}+2\sqrt{6}\,i-\sqrt{2}\,i^2$

$=(-6+\sqrt{2})+\sqrt{3}(1+2\sqrt{2})i.$

EXAMPLE 3 *Express each of the following in the form* $(a+ib)$:

(i) $(3+\sqrt{-5})(3-\sqrt{-5})$ *(ii)* $(-2+\sqrt{-3})(-3+2\sqrt{-3})$

(iii) $(-2+3i)^2$ *(iv)* $(\sqrt{5}-7i)^2$

SOLUTION We have

(i) $(3+\sqrt{-5})(3-\sqrt{-5}) = (3+\sqrt{5}\,i)(3-\sqrt{5}\,i) = \{3^2-(\sqrt{5}\,i)^2\}$

$= (9-5i^2) = (9+5) = 14+0i.$

$[\because\ (z_1+z_2)(z_1-z_2) = (z_1^2-z_2^2)]$

(ii) $(-2+\sqrt{-3})(-3+2\sqrt{-3}) = (-2+\sqrt{3}\,i)(-3+2\sqrt{3}\,i)$

$= (6-4\sqrt{3}\,i-3\sqrt{3}\,i+6i^2)$

$= -7\sqrt{3}\,i = 0-7\sqrt{3}\,i.$

(iii) $(-2+3i)^2 = (-2)^2+(3i)^2+2\times(-2)\times 3i$

$= (4+9i^2-12i) = (4-9-12i) = (-5-12i).$

(iv) $(\sqrt{5}-7i)^2 = (5)^2+(7i)^2-2\times\sqrt{5}\times 7i$

$= (25+49i^2-14\sqrt{5}\,i) = (25-49-14\sqrt{5}\,i)$

$= (-24-14\sqrt{5}\,i).$

EXAMPLE 4 *Prove that* $(1-i)^4 = -4.$

SOLUTION We have

$(1-i)^4 = (1-i)^2\times(1-i)^2 = (1+i^2-2i)\times(1+i^2-2i)$

$= (-2i)(-2i) = 4i^2 = 4\times(-1) = -4.$ $[\because\ i^2=-1]$

EXAMPLE 5 *Express* $(2+3i)^3$ *in the form* $(a+ib).$

SOLUTION We know that $(z_1+z_2)^3 = z_1^3+z_2^3+3z_1z_2(z_1+z_2).$

$\therefore\quad (2+3i)^3 = 2^3+(3i)^3+3\times 2\times 3i\times(2+3i)$

$= 8+27i^3+36i+54i^2$

$= (8-27i+36i-54)$ $[\because\ i^3=-i \text{ and } i^2=-1]$

$= (-46+9i).$

Hence, $(2+3i)^3 = (-46+9i).$

EXAMPLE 6 *Express* $(3-5i)^3$ *in the form* $(a+ib).$

SOLUTION We know that $(z_1-z_2)^3 = z_1^3-z_2^3-3z_1z_2(z_1-z_2).$

$\therefore\quad (3-5i)^3 = 3^3-(5i)^3-3\times 3\times 5i\times(3-5i)$

$= 27-125i^3-135i+225i^2$

$= (27+125i-135i-225)$ $[\because\ i^3=-i \text{ and } i^2=-1]$

$= (-198-10i).$

EXAMPLE 7 *If* $z = (\sqrt{2} - \sqrt{-3})$, *find* $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, $\bar{z}$ *and* $|z|$.

SOLUTION The given number may be written as

$$z = \sqrt{2} - i\sqrt{3}.$$

$$\therefore \quad \mathrm{Re}(z) = \sqrt{2}, \mathrm{Im}(z) = -\sqrt{3}, \bar{z} = \overline{(\sqrt{2} - i\sqrt{3})} = (\sqrt{2} + i\sqrt{3})$$

and $|z|^2 = \{(\sqrt{2})^2 + (-\sqrt{3})^2\} = (2+3) = 5 \Rightarrow |z| = \sqrt{5}$.

EXAMPLE 8 *Write down the modulus of:*

(i) $4 + \sqrt{-3}$ *(ii)* $2 - 5i$ *(iii)* $-i$ *(iv)* $(-1+3i)^2$

SOLUTION We know that, if $z = (a+ib)$ then $|z| = \sqrt{a^2 + b^2}$.

(i) Let $z = 4 + \sqrt{-3}$. Then, $z = 4 + i\sqrt{3}$.

$\therefore \quad |z|^2 = \{(4)^2 + (\sqrt{3})^2\} = (16+3) = 19 \Rightarrow |z| = \sqrt{19}$.

(ii) Let $z = 2 - 5i$. Then,

$$|z|^2 = \{2^2 + (-5)^2\} = (4+25) = 29 \Rightarrow |z| = \sqrt{29}.$$

(iii) Let $z = 0 - i$. Then,

$$|z|^2 = \{0^2 + (-1)^2\} = (0+1) = 1 \Rightarrow |z| = \sqrt{1} = 1.$$

(iv) Let $z = (-1+3i)^2$. Then,

$$z = (-1)^2 + (3i)^2 + 2 \times (-1) \times 3i = (-8 - 6i).$$

$\therefore \quad |z|^2 = \{(-8)^2 + (-6)^2\} = (64+36) = 100 \Rightarrow |z| = \sqrt{100} = 10.$

EXAMPLE 9 *Write down the conjugate of each of the following:*

(i) $(-5 + \sqrt{-1})$ *(ii)* $(-6 - \sqrt{-3})$ *(iii)* i^3 *(iv)* $(4+5i)^2$

SOLUTION We know that the conjugate of $z = (a+ib)$ is given as $\bar{z} = (a-ib)$.

(i) Let $z = -5 + \sqrt{-1} = (-5+i)$. Then,

$$\bar{z} = \overline{(-5+i)} = (-5-i).$$

(ii) Let $z = -6 - \sqrt{-3} = -6 - i\sqrt{3}$. Then,

$$\bar{z} = \overline{(-6 - i\sqrt{3})} = (-6 + i\sqrt{3}).$$

(iii) Let $z = i^3 = -i = 0 - i$. Then,

$$\bar{z} = \overline{(0-i)} = (0+i) = i.$$

(iv) Let $z = (4+5i)^2 = (4)^2 + (5i)^2 + 2 \times 4 \times 5i$

$$= (16 - 25 + 40i) = (-9 + 40i).$$

$\therefore \quad \bar{z} = \overline{(-9+40i)} = (-9 - 40i)$.

EXAMPLE 10 *Find the multiplicative inverse of each of the following:*

(i) $\sqrt{5} + 3i$ *(ii)* $4 - 3i$ *(iii)* $(3i-1)^2$ *(iv)* $-i$

SOLUTION We know that the multiplicative inverse of the complex number z is given by $z^{-1} = \dfrac{\bar{z}}{|z|^2}$.

(i) Let $z = (\sqrt{5} + 3i)$. Then,

$$\bar{z} = (\sqrt{5} - 3i) \text{ and } |z|^2 = (\sqrt{5})^2 + 3^2 = (5 + 9) = 14.$$

$$\therefore \quad z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{(\sqrt{5} - 3i)}{14} = \left(\frac{\sqrt{5}}{14} - \frac{3}{14}i\right).$$

(ii) Let $z = 4 - 3i$. Then,

$$\bar{z} = \overline{(4 - 3i)} = (4 + 3i) \text{ and } |z|^2 = \{(4)^2 + (-3)^2\} = (16 + 9) = 25.$$

$$\therefore \quad z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{(4 + 3i)}{25} = \left(\frac{4}{25} + \frac{3}{25}i\right).$$

(iii) Let $z = (3i - 1)^2 = (9i^2 + 1 - 6i) = (-9 + 1 - 6i) = (-8 - 6i)$.

$$\therefore \quad \bar{z} = \overline{(-8 - 6i)} = (-8 + 6i)$$

$$\text{and } |z|^2 = \{(-8)^2 + (-6)^2\} = (64 + 36) = 100.$$

$$\text{Hence, } z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{(-8 + 6i)}{100} = \left(\frac{-8}{100} + \frac{6}{100}i\right) = \left(\frac{-2}{25} + \frac{3}{50}i\right).$$

(iv) Let $z = (0 - i)$. Then,

$$\bar{z} = \overline{(0 - i)} = i \text{ and } |z|^2 = \sqrt{0^2 + (-1)^2} = 1.$$

$$\therefore \quad z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{i}{1} = i.$$

EXAMPLE 11 *Express each of the following in the form* $(a + ib)$:

$$(i)\ \frac{i}{(1+i)} \qquad (ii)\ (-1 + \sqrt{3}\,i)^{-1} \qquad (iii)\ \frac{5 + \sqrt{2}\,i}{1 - \sqrt{2}\,i}$$

SOLUTION We have

(i) $$\frac{i}{(1+i)} = \frac{i}{(1+i)} \times \frac{(1-i)}{(1-i)} = \frac{i(1-i)}{(1^2 - i^2)}$$

$$= \frac{(i - i^2)}{1 - (-1)} = \frac{(i+1)}{2} = \frac{(1+i)}{2} = \left(\frac{1}{2} + \frac{1}{2}i\right).$$

(ii) $$(-1 + \sqrt{3}\,i)^{-1} = \frac{1}{(-1 + \sqrt{3}\,i)} \times \frac{(-1 - \sqrt{3}\,i)}{(-1 - \sqrt{3}\,i)}$$

$$= \frac{(-1 - \sqrt{3}\,i)}{(-1 + \sqrt{3}\,i)(-1 - \sqrt{3}\,i)} = \frac{(-1 - \sqrt{3}\,i)}{(-1)^2 - (\sqrt{3}\,i)^2}$$

$$= \frac{(-1 - \sqrt{3}\,i)}{(1 - 3i^2)} = \frac{(-1 - \sqrt{3}\,i)}{4} = \left(\frac{-1}{4} - \frac{\sqrt{3}}{4}i\right).$$

(iii) $$\frac{5 + \sqrt{2}\,i}{1 - \sqrt{2}\,i} = \frac{(5 + \sqrt{2}\,i)}{(1 - \sqrt{2}\,i)} \times \frac{(1 + \sqrt{2}\,i)}{(1 + \sqrt{2}\,i)} = \frac{(5 + \sqrt{2}\,i)(1 + \sqrt{2}\,i)}{(1 - 2i^2)}$$

$$= \frac{(3 + 6\sqrt{2}\,i)}{3} = (1 + 2\sqrt{2}\,i).$$

EXAMPLE 12 *Reduce* $\left(\frac{1}{1+2i}+\frac{3}{1-i}\right)\left(\frac{3-2i}{1+3i}\right)$ *to the form* $(a+ib)$.

SOLUTION We have

$$\left(\frac{1}{1+2i}+\frac{3}{1-i}\right)\left(\frac{3-2i}{1+3i}\right)$$

$$=\left\{\frac{(1-i)+(3+6i)}{(1+2i)(1-i)}\right\}\left(\frac{3-2i}{1+3i}\right)=\frac{(4+5i)}{(3+i)}\times\frac{(3-2i)}{(1+3i)}$$

$$=\frac{(4+5i)(3-2i)}{(3+i)(1+3i)}=\frac{(12+10)+(15-8)i}{(3-3)+(9i+i)}=\frac{(22+7i)}{10i}\times\frac{i}{i}$$

$$=\frac{(7i^2+22i)}{10i^2}=\frac{-7+22i}{-10}=\left(\frac{7}{10}-\frac{22}{10}i\right)=\left(\frac{7}{10}-\frac{11}{5}i\right).$$

EXAMPLE 13 *Reduce* $\left(\frac{1+i}{1-i}-\frac{1-i}{1+i}\right)$ *to the form* $(a+ib)$ *and hence find its modulus.*

SOLUTION Let $z=\left(\frac{1+i}{1-i}-\frac{1-i}{1+i}\right)=\frac{(1+i)^2-(1-i)^2}{(1-i)(1+i)}=\frac{(4\times1\times i)}{(1-i^2)}=\frac{4i}{2}=2i.$

Thus, $z=0+2i \Rightarrow |z|=\sqrt{0^2+2^2}=\sqrt{4}=2.$

Hence, $z=0+2i$ and $|z|=2.$

EXAMPLE 14 *If* $\sqrt{\frac{1+i}{1-i}}=(a+ib)$ *then show that* $(a^2+b^2)=1.$

SOLUTION We have

$$(a+ib)=\sqrt{\frac{1+i}{1-i}}=\frac{\sqrt{1+i}}{\sqrt{1-i}}\times\frac{\sqrt{1+i}}{\sqrt{1+i}}=\frac{(1+i)}{\sqrt{1-i^2}}$$

$$=\frac{(1+i)}{\sqrt{2}}=\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)$$

$$\Rightarrow \quad |a+ib|^2=\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2=\left(\frac{1}{2}+\frac{1}{2}\right)=1.$$

$$\Rightarrow \quad (a^2+b^2)=1.$$

Hence, $(a^2+b^2)=1.$

EXAMPLE 15 *Find the least positive integer m for which* $\left(\frac{1+i}{1-i}\right)^m=1.$

SOLUTION We have

$$\left(\frac{1+i}{1-i}\right)=\frac{(1+i)}{(1-i)}\times\frac{(1+i)}{(1+i)}=\frac{(1+i)^2}{(1-i^2)}=\frac{(1+2i+i^2)}{2}=\frac{2i}{2}=i.$$

$$\therefore \quad \left(\frac{1+i}{1-i}\right)^m=1 \Rightarrow i^m=1.$$

And, we know that 4 is the least positive integer such that $i^4=1$ and therefore, $m=4.$

EXAMPLE 16 *Separate* $\left(\frac{3+\sqrt{-1}}{2-\sqrt{-1}}\right)$ *into real and imaginary parts and hence find its modulus.*

SOLUTION Let $z=\left(\frac{3+\sqrt{-1}}{2-\sqrt{-1}}\right)=\left(\frac{3+i}{2-i}\right)=\frac{(3+i)}{(2-i)}=\frac{(3+i)}{(2-i)}\times\frac{(2+i)}{(2+i)}$

$$=\frac{(3+i)(2+i)}{(2-i)(2+i)}=\frac{(6+i^2)+5i}{(4-i^2)}=\frac{5+5i}{\{4-(-1)\}}=\frac{5(1+i)}{5}=(1+i).$$

$\therefore \quad |z|=\sqrt{1^2+1^2}=\sqrt{2}.$

Hence, $z=(1+i)$ and $|z|=\sqrt{2}$.

EXAMPLE 17 *Reduce* $\left\{\frac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}-\sqrt{5-12i}}\right\}$ *to the form* $(a+ib)$ *and hence find its conjugate.*

SOLUTION We have

$$z=\frac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}-\sqrt{5-12i}}$$

$$=\frac{(\sqrt{5+12i}+\sqrt{5-12i})}{(\sqrt{5+12i}-\sqrt{5-12i})}\times\frac{(\sqrt{5+12i}+\sqrt{5-12i})}{(\sqrt{5+12i}+\sqrt{5-12i})}$$

$$=\frac{(\sqrt{5+12i}+\sqrt{5-12i})^2}{(5+12i)-(5-12i)}=\frac{(5+12i)+(5-12i)+2\sqrt{(5)^2-(12i)^2}}{24i}$$

$$=\frac{10+2\sqrt{25+144}}{24i}=\frac{10+2\sqrt{169}}{24i}=\frac{(10+2\times13)}{24i}$$

$$=\frac{36}{24i}=\frac{3}{2i}=\frac{3}{2i}\times\frac{i}{i}=\frac{3i}{2i^2}=-\frac{3}{2}i.$$

Hence, $z=\left(0-\frac{3}{2}i\right)$ and $\bar{z}=\overline{\left(0-\frac{3}{2}i\right)}=\left(0+\frac{3}{2}i\right)$.

EXAMPLE 18 *If* $(x+iy)^{1/3}=(a+ib)$ *then prove that:*

(i) $\frac{x}{a}+\frac{y}{b}=4(a^2-b^2)$ *(ii)* $\frac{x}{a}-\frac{y}{b}=-2(a^2+b^2)$

SOLUTION We have

$$(x+iy)^{1/3}=(a+ib)$$

$\Rightarrow \quad (x+iy)=(a+ib)^3$ [on cubing both sides]

$\Rightarrow \quad (x+iy)=a^3+i^3b^3+3iab(a+ib)$

$$=a^3-ib^3+3a^2bi-3ab^2=(a^3-3ab^2)+i(3a^2b-b^3)$$

$\Rightarrow \quad x=a^3-3ab^2$ and $y=3a^2b-b^3$

[on equating real and imaginary parts separately]

$$\Rightarrow \quad \frac{x}{a} = (a^2 - 3b^2) \text{ and } \frac{y}{b} = (3a^2 - b^2)$$

$$\Rightarrow \quad \left(\frac{x}{a} + \frac{y}{b}\right) = 4(a^2 - b^2) \text{ and } \left(\frac{x}{a} - \frac{y}{b}\right) = -2(a^2 + b^2).$$

Hence, (i) $\frac{x}{a} + \frac{y}{b} = 4(a^2 - b^2)$ and (ii) $\frac{x}{a} - \frac{y}{b} = -2(a^2 + b^2)$.

EXAMPLE 19 *Let z be a complex number such that $z \neq 1$ and $|z| = 1$. Then, prove that $\left(\frac{z-1}{z+1}\right)$ is purely imaginary. What will be your conclusion when $z = 1$?*

SOLUTION Let $z = (x + iy)$ be the given complex number such that $z \neq 1$ and $|z| = 1$. Then,

$$|z| = 1 \Rightarrow |z|^2 \Rightarrow x^2 + y^2 = 1 \Rightarrow x^2 + y^2 - 1 = 0. \qquad \ldots \text{(i)}$$

$$\therefore \quad \left(\frac{z-1}{z+1}\right) = \frac{(z-1)}{(z+1)} = \frac{x + iy - 1}{x + iy + 1} = \frac{(x-1) + iy}{(x+1) + iy} \times \frac{(x+1) - iy}{(x+1) - iy}$$

$$= \frac{\{(x-1) + iy\}\{(x+1) - iy\}}{\{(x+1)^2 + y^2\}}$$

$$= \frac{(x^2 - 1 + y^2) + i\{(x+1)y - (x-1)y\}}{\{(x+1)^2 + y^2\}}$$

$$= \frac{(x^2 + y^2 - 1) + (2y)i}{\{(x+1)^2 + y^2\}}$$

$$= \left\{\frac{2yi}{(x+1)^2 + y^2}\right\} \quad \text{[using (i)],}$$

which is purely imaginary.

Particular Case *When $z = 1$.*

In this case, $z = 1 \Rightarrow x + iy = 1$

$$\Rightarrow (x-1) + iy = 0$$

$$\Rightarrow x - 1 = 0, y = 0 \Rightarrow x = 1, y = 0.$$

$$\therefore \quad \frac{z-1}{z+1} = \frac{(x+iy)-1}{(x+iy)+1} = \frac{(x-1)+iy}{(x+1)+iy} = \frac{(1-1)+i\times 0}{(1+1)+i\times 0} = \frac{0}{2} = 0.$$

Thus, $z = 1 \Rightarrow \frac{z-1}{z+1}$ is purely real.

EXAMPLE 20 *Find the real value of θ for which $\left(\frac{3 + 2i\sin\theta}{1 - 2i\sin\theta}\right)$ is purely real.*

SOLUTION We have

$$\left(\frac{3 + 2i\sin\theta}{1 - 2i\sin\theta}\right) = \frac{(3 + 2i\sin\theta)}{(1 - 2i\sin\theta)} \times \frac{(1 + 2i\sin\theta)}{(1 + 2i\sin\theta)}$$

$$= \frac{(3 + 2i\sin\theta)(1 + 2i\sin\theta)}{(1 - 4i^2\sin^2\theta)}$$

$$= \frac{(3 - 4\sin^2\theta) + i(6\sin\theta + 2\sin\theta)}{(1 + 4\sin^2\theta)}$$

$$= \frac{(3 - 4\sin^2\theta) + i(8\sin\theta)}{(1 + 4\sin^2\theta)}.$$

Now, $\left(\frac{3 + 2i\sin\theta}{1 - 2i\sin\theta}\right)$ will be purely real only when $\frac{8\sin\theta}{(1 + 4\sin^2\theta)} = 0.$

This happens only when $8\sin\theta = 0 \Leftrightarrow \sin\theta = 0 \Leftrightarrow \theta = n\pi, n \in N.$

Hence, the required value of θ is $n\pi$, where $n \in N.$

EXAMPLE 21 *Show that* $|1 - i|^x = 2^x$ *has no nonzero integral solution.*

SOLUTION $|1 - i|^x = 2^x \Rightarrow (\sqrt{2})^x = 2^x$ $[\because |1 - i| = \sqrt{1^2 + (-1)^2} = \sqrt{2}]$

$$\Rightarrow 2^{x/2} = 2^x \Rightarrow \frac{2^x}{2^{x/2}} = 1$$

$$\Rightarrow 2^{x/2} = 1 = 2^0 \Rightarrow \frac{x}{2} = 0 \Rightarrow x = 0.$$

Thus, $x = 0$ is the only solution of the given equation.

Hence, the given equation has no nonzero integral solution.

EXAMPLE 22 *Solve for x and y:*

(i) $(3x - 7) + 2iy = -5y + (5 + x)i$

(ii) $(x + iy)(2 - 3i) = 4 + i$

SOLUTION (i) The given equation is

$$(3x - 7) + 2iy = -5y + (5 + x)i.$$

Equating the real parts and the imaginary parts of the given equation separately, we get

$3x - 7 = -5y$ and $2y = 5 + x$

$\Rightarrow$ $3x + 5y = 7$ and $x - 2y = -5$

$\Rightarrow$ $x = -1$ and $y = 2$ [on solving $3x + 5y = 7, x - 2y = -5$].

Hence, $x = -1$ and $y = 2.$

(ii) The given equation is

$(x + iy)(2 - 3i) = 4 + i$

$\Rightarrow$ $(2x + 3y) + i(2y - 3x) = 4 + i$

$\Rightarrow$ $2x + 3y = 4$ and $2y - 3x = 1$

[equating real parts and imaginary parts separately]

$\Rightarrow$ $2x + 3y = 4$ and $-3x + 2y = 1$

$\Rightarrow$ $x = \frac{5}{13}$ and $y = \frac{14}{13}$ [on solving $2x + 3y = 4, -3x + 2y = 1$].

Hence, $x = \frac{5}{13}$ and $y = \frac{14}{13}$ is the required solution.

EXAMPLE 23 *Find the real values of x and y for which* $\left(\frac{x-1}{3+i}+\frac{y-1}{3-i}\right)=i.$

SOLUTION We have

$$\frac{x-1}{3+i}+\frac{y-1}{3-i}=i$$

$$\Rightarrow \quad \frac{(x-1)}{(3+i)}\times\frac{(3-i)}{(3-i)}+\frac{(y-1)}{(3-i)}\times\frac{(3+i)}{(3+i)}=i$$

$$\Rightarrow \quad \frac{(x-1)(3-i)}{(9-i^2)}+\frac{(y-1)(3+i)}{(9-i^2)}=i$$

$$\Rightarrow \quad \frac{3(x-1)-i(x-1)}{10}+\frac{3(y-1)+i(y-1)}{10}=i$$

$$\Rightarrow \quad 3(x-1)-i(x-1)+3(y-1)+i(y-1)=10i$$

$$\Rightarrow \quad (3x-3+3y-3)+i(y-1-x+1)=10i$$

$$\Rightarrow \quad (3x+3y-6)+i(y-x)=10i$$

$$\Rightarrow \quad 3(x+y-2)=0 \text{ and } y-x=10$$

[equating real parts and the imaginary parts separately]

$$\Rightarrow \quad x+y-2=0 \text{ and } y-x=10$$

$$\Rightarrow \quad x+y=2 \text{ and } -x+y=10$$

$$\Rightarrow \quad x=-4 \text{ and } y=6 \qquad \text{[on solving } x+y=2, -x+y=10\text{]}.$$

Hence, $x=-4$ and $y=6$ are the required values.

EXAMPLE 24 *Find the complex number z for which* $|z+1|=z+2(1+i)$.

SOLUTION Let the required complex number be $z=(x+iy)$. Then,

$$|z+1|=z+2(1+i)$$

$$\Rightarrow \quad |(x+iy)+1|=(x+iy)+2(1+i)$$

$$\Rightarrow \quad \sqrt{(x+1)^2+y^2}=(x+2)+i(y+2)$$

$$\Rightarrow \quad \sqrt{(x+1)^2+y^2}=(x+2) \text{ and } y+2=0$$

[equating real parts and imaginary parts separately]

$$\Rightarrow \quad y=-2 \text{ and } \sqrt{(x+1)^2+(-2)^2}=(x+2)$$

$$\Rightarrow \quad y=-2 \text{ and } \sqrt{x^2+2x+5}=(x+2)$$

$$\Rightarrow \quad y=-2 \text{ and } (x^2+2x+5)=(x+2)^2$$

$$\Rightarrow \quad x^2+2x+5=x^2+4x+4 \text{ and } y=-2$$

$$\Rightarrow \quad 2x=1 \text{ and } y=-2 \;\Rightarrow\; x=\frac{1}{2} \text{ and } y=-2.$$

Hence, the required complex number is $z=\left(\frac{1}{2}-2i\right)$.

EXAMPLE 25 *Solve the equation* $|z| + z = (2 + i)$ *for complex value of z.*

SOLUTION Let $z = (x + iy)$. Then,

$$|z| + z = 2 + i$$

$$\Rightarrow \quad |x + iy| + (x + iy) = 2 + i$$

$$\Rightarrow \quad \{\sqrt{x^2 + y^2} + x\} + iy = (2 + i)$$

$$\Rightarrow \quad \sqrt{x^2 + y^2} + x = 2 \text{ and } y = 1$$

[equating real parts and imaginary parts separately]

$$\Rightarrow \quad y = 1 \text{ and } \sqrt{x^2 + 1^2} + x = 2 \;\Rightarrow\; y = 1 \text{ and } \sqrt{x^2 + 1} = (2 - x)$$

$$\Rightarrow \quad y = 1 \text{ and } x^2 + 1 = (2 - x)^2 \;\Rightarrow\; y = 1 \text{ and } x^2 + 1 = x^2 - 4x + 4$$

$$\Rightarrow \quad 4x = 3 \text{ and } y = 1 \;\Rightarrow\; x = \frac{3}{4} \text{ and } y = 1.$$

Hence, $z = \left(\frac{3}{4} + i\right)$ is the desired solution.

EXAMPLE 26 *Solve the equation* $z + \sqrt{2}|z + 1| + i = 0$ *for complex value of z.*

SOLUTION Let the required complex number be $z = x + iy$. Then,

$$z + \sqrt{2}|z + 1| + i = 0$$

$$\Rightarrow \quad (x + iy) + \sqrt{2}|(x + iy) + 1| + i = 0$$

$$\Rightarrow \quad x + \sqrt{2}\{\sqrt{(x + 1)^2 + y^2}\} + (y + 1)i = 0$$

$$\Rightarrow \quad x + \sqrt{2}\{\sqrt{(x + 1)^2 + y^2}\} = 0 \text{ and } y + 1 = 0$$

[equating real parts and imaginary parts separately on both sides]

$$\Rightarrow \quad y = -1 \text{ and } \sqrt{2}\{\sqrt{(x + 1)^2 + (-1)^2}\} = (-x)$$

$$\Rightarrow \quad y = -1 \text{ and } \sqrt{2} \cdot \sqrt{x^2 + 2x + 2} = (-x)$$

$$\Rightarrow \quad y = -1 \text{ and } 2(x^2 + 2x + 2) = x^2$$

$$\Rightarrow \quad y = -1 \text{ and } x^2 + 4x + 4 = 0 \;\Rightarrow\; (x + 2)^2 = 0 \text{ and } y = -1$$

$$\Rightarrow \quad x + 2 = 0 \text{ and } y = -1 \;\Rightarrow\; x = -2 \text{ and } y = -1.$$

Hence, the required complex number is $z = (-2 - i)$.

EXAMPLE 27 *If* $z = 3 + 2i$, *prove that* $z^2 - 6z + 13 = 0$ *and hence deduce that* $3z^3 - 13z^2 + 9z + 65 = 0$.

SOLUTION $z = 3 + 2i \;\Rightarrow\; z - 3 = 2i \;\Rightarrow\; (z - 3)^2 = 4i^2$

$$\Rightarrow \quad z^2 - 6z + 13 = 0. \qquad \ldots \text{(i)}$$

Thus, $z^2 - 6z + 13 = 0$.

Now, $3z^3 - 13z^2 + 9z + 65 = 3z(z^2 - 6z + 13) + 5(z^2 - 6z + 13)$

$= (3z \times 0) + (5 \times 0) = 0$ [using (i)].

Hence, $3z^3 - 13z^2 + 9z + 65 = 0$.

EXAMPLE 28 *If* $z = -5 + 3i$, *find the value of* $(z^4 + 9z^3 + 26z^2 - 14z + 8)$.

SOLUTION We have

$$z = -5 + 3i \Rightarrow (z + 5) = 3i$$

$$\Rightarrow (z+5)^2 = 9i^2 \Rightarrow z^2 + 10z + 34 = 0. \quad \ldots \text{(i)}$$

Now, $z^4 + 9z^3 + 26z^2 - 14z + 8$

$$= z^2(z^2 + 10z + 34) - z(z^2 + 10z + 34) + 2(z^2 + 10z + 34) - 60$$

$$= (z^2 \times 0) - (z \times 0) + (2 \times 0) - 60 = -60 \quad \text{[using (i)]}.$$

Hence, the value of $z^4 + 9z^3 + 26z^2 - 14z + 8$ is –60.

EXAMPLE 29 *If* $z = 2 + i$, *prove that* $z^3 + 3z^2 - 9z + 8 = (1 + 14i)$.

SOLUTION We have

$$z = 2 + i \Rightarrow z - 2 = i$$

$$\Rightarrow (z-2)^2 = i^2 \Rightarrow z^2 - 4z + 5 = 0. \quad \ldots \text{(i)}$$

$\therefore$ $z^3 + 3z^2 - 9z + 8$

$$= z(z^2 - 4z + 5) + 7(z^2 - 4z + 5) + 14z - 27$$

$$= (z \times 0) + (7 \times 0) + 14z - 27 = (14z - 27) \quad \text{[using (i)]}$$

$$= 14(2 + i) - 27 = (1 + 14i).$$

Hence, $z^3 + 3z^2 - 9z + 8 = (1 + 14i)$.

EXERCISE 5B

1. Simplify each of the following and express it in the form $a + ib$:

(i) $2(3 + 4i) + i(5 - 6i)$ (ii) $(3 + \sqrt{-16}) - (4 - \sqrt{-9})$

(iii) $(-5 + 6i) - (-2 + i)$ (iv) $(8 - 4i) - (-3 + 5i)$

(v) $(1 - i)^2(1 + i) - (3 - 4i)^2$ (vi) $(5 + \sqrt{-3})(5 - \sqrt{-3})$

(vii) $(3 + 4i)(2 - 3i)$ (viii) $(-2 + \sqrt{-3})(-3 + 2\sqrt{-3})$

2. Simplify and express each of the following in the form $(a + ib)$:

(i) $(2 + \sqrt{-3})^2$ (ii) $(5 - 2i)^2$ (iii) $(-3 + 5i)^3$

(iv) $\left(-2 - \frac{1}{3}i\right)^3$ (v) $(4 - 3i)^{-1}$ (vi) $(-2 + \sqrt{-3})^{-1}$

(vii) $(2 + i)^{-2}$ (viii) $(1 + 2i)^{-3}$ (ix) $(1 + i)^3 - (1 - i)^3$

3. Express each of the following in the form $(a + ib)$:

(i) $\frac{1}{(4 + 3i)}$ (ii) $\frac{(3 + 4i)}{(4 - 5i)}$ (iii) $\frac{(5 + \sqrt{2}\,i)}{(1 - \sqrt{2}\,i)}$

(iv) $\frac{(-2 + 5i)}{(3 - 6i)}$ (v) $\frac{(3 - 4i)}{(4 - 2i)(1 + i)}$ (vi) $\frac{(3 - 2i)(2 + 3i)}{(1 + 2i)(2 - i)}$

(vii) $\frac{(2 + 3i)^2}{(2 - i)}$ (viii) $\frac{(1 - i)^3}{(1 - i^3)}$ (ix) $\frac{(1 + 2i)^3}{(1 + i)(2 - i)}$

4. Simplify and express each of the following in the form $(a+ib)$:

(i) $\left(\frac{5}{-3+2i}+\frac{2}{1-i}\right)\left(\frac{4-5i}{3+2i}\right)$ (ii) $\left(\frac{1}{1-4i}-\frac{2}{1+i}\right)\left(\frac{1-i}{5+3i}\right)$

5. Show that

(i) $\left\{\frac{(3+2i)}{(2-3i)}+\frac{(3-2i)}{(2+3i)}\right\}$ is purely real,

(ii) $\left\{\frac{(\sqrt{7}+i\sqrt{3})}{(\sqrt{7}-i\sqrt{3})}+\frac{(\sqrt{7}-i\sqrt{3})}{(\sqrt{7}+i\sqrt{3})}\right\}$ is purely real.

6. Find the real values of θ for which $\left(\frac{1+i\cos\theta}{1-2i\cos\theta}\right)$ is purely real.

7. If $|z+i| = |z-i|$, prove that z is real.

8. Give an example of two complex numbers z_1 and z_2 such that $z_1 \neq z_2$ and $|z_1| = |z_2|$.

9. Find the conjugate of each of the following:

(i) $(-5-2i)$ (ii) $\frac{1}{(4+3i)}$ (iii) $\frac{(1+i)^2}{(3-i)}$ (iv) $\frac{(1+i)(2+i)}{(3+i)}$

(v) $\sqrt{-3}$ (vi) $\sqrt{2}$ (vii) $-\sqrt{-1}$ (viii) $(2-5i)^2$

10. Find the modulus of each of the following:

(i) $(3+\sqrt{-5})$ (ii) $(-3-4i)$ (iii) $(7+24i)$ (iv) $3i$

(v) $\frac{(3+2i)^2}{(4-3i)}$ (vi) $\frac{(2-i)(1+i)}{(1+i)}$ (vii) 5 (viii) $(1+2i)(i-1)$

11. Find the multiplicative inverse of each of the following:

(i) $(1-\sqrt{3}\,i)$ (ii) $(2+5i)$ (iii) $\frac{(2+3i)}{(1+i)}$ (iv) $\frac{(1+i)(1+2i)}{(1+3i)}$

12. If $\left(\frac{1-i}{1+i}\right)^{100} = (a+ib)$, find the values of a and b.

13. If $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x+iy$, find x and y.

14. If $(x+iy) = \frac{a+ib}{a-ib}$, prove that $x^2+y^2 = 1$.

15. If $(a+ib) = \frac{c+i}{c-i}$, where c is real, prove that $a^2+b^2 = 1$ and $\frac{b}{a} = \frac{2c}{c^2-1}$.

16. Show that $(1-i)^n\left(1-\frac{1}{i}\right)^n = 2^n$ for all $n \in N$.

17. Find the smallest positive integer n for which $(1+i)^{2n} = (1-i)^{2n}$.

18. Prove that $(x+1+i)(x+1-i)(x-1+i)(x-1-i) = (x^4+4)$.

19. If $a = (\cos\theta + i\sin\theta)$, prove that $\frac{1+a}{1-a} = \left(\cot\frac{\theta}{2}\right)i$.

20. If $z_1 = (2-i)$ and $z_2 = (1+i)$, find $\left|\frac{z_1+z_2+1}{z_1-z_2+i}\right|$.

21. Find the real values of x and y for which:

(i) $(1-i)x+(1+i)y = 1-3i$ (ii) $(x+iy)(3-2i) = (12+5i)$

(iii) $x+4yi = ix+y+3$ (iv) $(1+i)y^2+(6+i) = (2+i)x$

(v) $\frac{(x+3i)}{(2+iy)} = (1-i)$ (vi) $\frac{(1+i)x-2i}{(3+i)}+\frac{(2-3i)y+i}{(3-i)} = i$

22. Find the real values of x and y for which $(x-iy)(3+5i)$ is the conjugate of $(-6-24i)$.

23. Find the real values of x and y for which the complex numbers $(-3+iyx^2)$ and (x^2+y+4i) are conjugates of each other.

24. If $z = (2-3i)$, prove that $z^2-4z+13 = 0$ and hence deduce that $4z^3-3z^2+169 = 0$.

25. If $(1+i)z = (1-i)\bar{z}$ then prove that $z = -i\bar{z}$.

26. If $\left(\frac{z-1}{z+1}\right)$ is purely an imaginary number and $z \neq -1$ then find the value of $|z|$.

27. Solve the system of equations, $\mathrm{Re}(z^2) = 0$, $|z| = 2$.

28. Find the complex number z for which $|z| = z+1+2i$.

ANSWERS (EXERCISE 5B)

1. (i) $(12+13i)$ (ii) $(-1+7i)$ (iii) $(-3+5i)$ (iv) $(11-9i)$

(v) $(9+22i)$ (vi) 28 (vii) $(18-i)$ (viii) $-7\sqrt{3}i$

2. (i) $(1+4\sqrt{3}i)$ (ii) $(21-20i)$ (iii) $(198+10i)$ (iv) $\left(\frac{-22}{3}-\frac{107}{27}i\right)$

(v) $\left(\frac{4}{25}+\frac{3}{25}i\right)$ (vi) $\left(\frac{-2}{7}-\frac{\sqrt{3}}{7}i\right)$ (vii) $\left(\frac{3}{25}-\frac{4}{25}i\right)$ (viii) $\left(\frac{-11}{125}+\frac{2}{125}i\right)$

(ix) $4i$

3. (i) $\left(\frac{4}{25}-\frac{3}{25}i\right)$ (ii) $\left(\frac{-8}{41}+\frac{31}{41}i\right)$ (iii) $(1+2\sqrt{2}i)$ (iv) $\left(\frac{-4}{5}+\frac{1}{15}i\right)$

(v) $\left(\frac{1}{4}-\frac{3}{4}i\right)$ (vi) $\left(\frac{63}{25}-\frac{16}{25}i\right)$ (vii) $\left(\frac{-22}{5}+\frac{19}{5}i\right)$ (viii) $(-2+0i)$

(ix) $\left(\frac{-7}{2}+\frac{1}{2}i\right)$

4. (i) $\left(\frac{5}{13}+\frac{4}{13}i\right)$ (ii) $\left(\frac{4}{17}+\frac{5}{17}i\right)$

6. $\theta = (2n+1)\frac{\pi}{2}, n \in I$

9. (i) $(-5+2i)$ (ii) $\left(\frac{4}{25}+\frac{3}{25}i\right)$ (iii) $\left(\frac{-1}{5}-\frac{3}{5}i\right)$ (iv) $\left(\frac{3}{5}-\frac{4}{5}i\right)$

(v) $-\sqrt{3}i$ (vi) $\sqrt{2}$ (vii) i (viii) $(-21+20i)$

10. (i) $\sqrt{14}$ (ii) 5 (iii) 25 (iv) 3

(v) $\frac{13}{5}$ (vi) $\sqrt{5}$ (vii) 5 (viii) $\sqrt{10}$

11. (i) $\left(\frac{1}{4}, \frac{\sqrt{3}}{4}i\right)$ (ii) $\left(\frac{2}{29} - \frac{5}{29}i\right)$ (iii) $\left(\frac{5}{13} - \frac{1}{13}i\right)$ (iv) $\left(\frac{4}{5} - \frac{3}{5}i\right)$

12. $a = 1, b = 0$ **13.** $x = 0, y = -2$ **17.** $n = 2$ **20.** $2\sqrt{2}$

21. (i) $x = 2, y = -1$ (ii) $x = 2, y = 3$ (iii) $x = 4, y = 1$

(iv) $(x = 5, y = 2)$ or $(x = 5, y = -2)$ (v) $x = 7, y = 5$ (vi) $x = 3, y = -1$

22. $x = 3, y = -3$ **23.** $(x = 1, y = -4)$ or $(x = -1, y = -4)$

26. $|z| = 1$ **27.** $z = \sqrt{2}(1 \pm i)$ or $z = \sqrt{2}(-1 \pm i)$ **28.** $\left(\frac{3}{2} - 2i\right)$

HINTS TO SOME SELECTED QUESTIONS

2. (iii) $(-3+5i)^3 = (5i-3)^3 = (5i)^3 - 3^3 - (3\times 5i\times 3)(5i-3)$

$= (-125i - 27 + 225 + 135i) = (198 + 10i)$.

(iv) $\left(-2 - \frac{1}{3}i\right)^3 = (-1)^3\left(2 + \frac{1}{3}i\right)^3 = -\left\{8 + \frac{1}{27}i^3 + 2i\left(2 + \frac{1}{3}i\right)\right\}$.

(v) $(4-3i)^{-1} = \frac{1}{(4-3i)} \times \frac{(4+3i)}{(4+3i)} = \frac{(4+3i)}{25}$.

(vi) $(-2+\sqrt{3}i)^{-1} = \frac{1}{(-2+\sqrt{3}i)} \times \frac{(-2-\sqrt{3}i)}{(-2-\sqrt{3}i)}$.

(vii) $(2+i)^{-2} = \frac{1}{(2+i)^2} = \frac{1}{(4+i^2+4i)} = \frac{1}{(3+4i)} \times \frac{(3-4i)}{(3-4i)}$.

(viii) $(1+2i)^{-3} = \frac{1}{(1+2i)^3} = \frac{1}{\{1+8i^3+6i(1+2i)\}} = \frac{1}{(-11-2i)} \times \frac{(-11+2i)}{(-11+2i)}$.

4. (i) $\left(\frac{5}{-3+2i} + \frac{2}{1-i}\right)\left(\frac{4-5i}{3+2i}\right) = \frac{5(1-i)+2(-3+2i)}{(-3+2i)(1-i)} \times \frac{(4-5i)}{(3+2i)}$

$= \frac{(-1-i)(4-5i)}{(-1+5i)(3+2i)} = \frac{(1+i)(4-5i)}{(1-5i)(3+2i)}$

$= \frac{(9-i)}{13(1-i)} \times \frac{(1+i)}{(1+i)} = \frac{(10+8i)}{26} = \left(\frac{5}{13} + \frac{4}{13}i\right)$.

5. (i) Given expression $= \frac{(3+2i)(2+3i)+(3-2i)(2-3i)}{(2-3i)(2+3i)} = \frac{0}{13} = 0$.

(ii) Given expression $= \frac{(\sqrt{7}+i\sqrt{3})^2 + (\sqrt{7}-i\sqrt{3})^2}{(\sqrt{7}-i\sqrt{3})(\sqrt{7}+i\sqrt{3})} = \frac{2\{(\sqrt{7})^2 + (i\sqrt{3})^2\}}{(7+3)} = \frac{8}{10} = \frac{4}{5}$.

6. Given expression $= \frac{(1+i\cos\theta)}{(1-2i\cos\theta)} \times \frac{(1+2i\cos\theta)}{(1+2i\cos\theta)} = \frac{(1-2\cos^2\theta)+i(3\cos\theta)}{(1+4\cos^2\theta)}$.

This expression will be purely real when $3\cos\theta = 0$.

Now, $3\cos\theta = 0 \Leftrightarrow \cos\theta = 0 \Leftrightarrow \theta = (2n+1)\frac{\pi}{2}$, where $n \in I$.

7. Let $z = (x + iy)$. Then,

$|z + i|^2 = |z - i|^2$

$\Rightarrow \quad |(x + iy) + i|^2 = |(x + iy) - i|^2$

$\Rightarrow \quad |x + (y + 1)i|^2 = |x + (y - 1)i|^2 \Rightarrow x^2 + (y + 1)^2 = x^2 + (y - 1)^2$

$\Rightarrow \quad (y + 1)^2 - (y - 1)^2 = 0 \Rightarrow 4y = 0 \Rightarrow y = 0.$

Hence, z is purely real.

8. Let $z_1 = (3 + 2i)$ and $z_2 = (3 - 2i)$. Then, $z_1 \neq z_2$.

But, $|z| = |z_2| = \sqrt{13}$.

10. (v) $z = \dfrac{(3+2i)^2}{(4-3i)} = \dfrac{(9+4i^2+12i)}{(4-3i)} = \dfrac{(5+12i)}{(4-3i)} \times \dfrac{(4+3i)}{(4+3i)} = \dfrac{-16+63i}{25}$

$\Rightarrow \quad |z|^2 = \left\{\left(\dfrac{-16}{25}\right)^2 + \left(\dfrac{63}{25}\right)^2\right\} = \left(\dfrac{256}{625} + \dfrac{3969}{625}\right) = \dfrac{4225}{625}$

$\Rightarrow \quad |z| = \sqrt{\dfrac{4225}{625}} = \dfrac{65}{25} = \dfrac{13}{5}.$

(vi) $z = \dfrac{(2-i)(1+i)}{(1-i)} = \dfrac{(3+i)}{(1-i)} \times \dfrac{(1+i)}{(1+i)} = \dfrac{(2+4i)}{2} = (1+2i).$

11. Use the formula, $z^{-1} = \dfrac{\bar{z}}{|z|^2}.$

(iii) $z = \dfrac{(2+3i)}{(1+i)} \times \dfrac{(1-i)}{(1-i)} = \dfrac{(5+i)}{2} = \left(\dfrac{5}{2} + \dfrac{1}{2}i\right).$

$|z|^2 = \left(\dfrac{25}{4} + \dfrac{1}{4}\right) = \dfrac{26}{4} = \dfrac{13}{2}$ and $\bar{z} = \left(\dfrac{5}{2} - \dfrac{1}{2}i\right) = \dfrac{(5-i)}{2}.$

$\therefore \quad z^{-1} = \dfrac{\bar{z}}{|z|^2} = \dfrac{(5-i)}{2} \times \dfrac{2}{13} = \dfrac{(5-i)}{13} = \left(\dfrac{5}{13} - \dfrac{1}{13}i\right).$

(iv) $z = \dfrac{(1+i)(1+2i)}{(1+3i)} = \dfrac{(-1+3i)}{(1+3i)} \times \dfrac{(1-3i)}{(1-3i)} = \dfrac{(8+6i)}{10} = \left(\dfrac{4}{5} + \dfrac{3}{5}i\right).$

$\therefore \quad \bar{z} = \left(\dfrac{4}{5} - \dfrac{3}{5}i\right) = \dfrac{(4-3i)}{5}$ and $|z|^2 = \left(\dfrac{16}{25} + \dfrac{9}{25}\right) = \dfrac{25}{25} = 1.$

$\therefore \quad z^{-1} = \dfrac{\bar{z}}{|z|^2} = \left(\dfrac{4}{5} - \dfrac{3}{5}i\right).$

12. $\dfrac{(1-i)}{(1+i)} = \dfrac{(1-i)}{(1+i)} \times \dfrac{(1-i)}{(1-i)} = \dfrac{(1-i)^2}{2} = \dfrac{(1+i^2-2i)}{2} = -i.$

$\therefore \quad \left(\dfrac{1-i}{1+i}\right)^{100} = (-i)^{100} = \{(-i)^4\}^{25} = (1)^{25} = 1 \qquad [\because (-i)^4 = i^4 = 1].$

Hence, $a = 1$ and $b = 0$.

13. $\left(\dfrac{1+i}{1-i}\right) = \dfrac{(1+i)}{(1-i)} \times \dfrac{(1+i)}{(1+i)} = \dfrac{(1+i)^2}{2} = \dfrac{(1+i^2+2i)}{2} = \dfrac{2i}{2} = i.$

Similarly, $\left(\dfrac{1-i}{1+i}\right) = -i.$

Given expression $= i^3 - (-i)^3 = 2i^3 = -2i = 0 + (-2)i.$

14. $(x + iy) = \dfrac{a+ib}{a-ib} \Rightarrow \overline{(x+iy)} = \dfrac{\overline{(a+ib)}}{\overline{(a-ib)}} \Rightarrow (x - iy) = \dfrac{(a-ib)}{(a+ib)}.$

$\therefore \quad (x+iy)(x-iy) = \dfrac{(a+ib)}{(a-ib)} \times \dfrac{(a-ib)}{(a+ib)} = 1 \Rightarrow x^2+y^2 = 1.$

15. $(a+ib) = \dfrac{(c+i)}{(c-i)} \times \dfrac{(c+i)}{(c+i)} = \dfrac{(c+i)^2}{(c^2+1)} = \dfrac{c^2+i^2+2ci}{(c^2+1)} = \dfrac{(c^2-1)}{(c^2+1)} + \dfrac{2c}{(c^2+1)}i.$

$\Rightarrow \quad |a+ib|^2 = \left\{\left(\dfrac{c^2-1}{c^2+1}\right)^2 + \dfrac{4c^2}{(c^2+1)^2}\right\} = \dfrac{(c^2-1)^2+4c^2}{(c^2+1)^2} = \dfrac{(c^2+1)^2}{(c^2+1)^2} = 1.$

$\Rightarrow \quad a^2+b^2 = 1.$

Now, $a = \dfrac{(c^2-1)}{(c^2+1)}$ and $b = \dfrac{2c}{(c^2+1)} \Rightarrow \dfrac{b}{a} = \dfrac{2c}{(c^2-1)}.$

16. $\left(1-\dfrac{1}{i}\right) = \dfrac{(i-1)}{i} \times \dfrac{i}{i} = \dfrac{(i^2-i)}{i^2} = \dfrac{(-1-i)}{-1} = (1+i).$

$\therefore \quad$ given expression $= (1-i)^n \times (1+i)^n = \{(1-i)+(1+i)\}^2 = (1-i^2)^n = 2^n.$

17. $\dfrac{(1+i)}{(1-i)} = \dfrac{(1+i)}{(1-i)} \times \dfrac{(1+i)}{(1+i)} = \dfrac{(1+i)^2}{2} = \dfrac{(1+i^2+2i)}{2} = \dfrac{2i}{2} = i.$

$\therefore \quad (1+i)^{2n} = (1-i)^{2n} \Rightarrow \dfrac{(1+i)^{2n}}{(1-i)^{2n}} = 1 \Rightarrow \left(\dfrac{1+i}{1-i}\right)^{2n} = 1 \Rightarrow i^{2n} = 1.$

Least value of $2n$ is 4 and so the least value of n is 2.

18. Given expression $= \{(x+1)+i\}\{(x+1)-i\}\{(x-1)+i\}\{(x-1)-i\}$

$= \{(x+1)^2-i^2\}\{(x-1)^2-i^2\} = \{(x+1)^2+1\}\{(x-1)^2+1\}$

$= \{(x^2+2)+2x\}\{(x^2+2)-2x\}$

$= (x^2+2)^2-4x^2 = (x^4+4).$

19. $\dfrac{1+a}{1-a} = \dfrac{(1+\cos\theta)+i\sin\theta}{(1-\cos\theta)-i\sin\theta} \times \dfrac{(1-\cos\theta)+i\sin\theta}{(1-\cos\theta)+i\sin\theta}$

$= \dfrac{(1-\cos^2\theta-\sin^2\theta)+2i\sin\theta}{(1-\cos\theta)^2+\sin^2\theta} = \dfrac{1-(\cos^2\theta+\sin^2\theta)+2i\sin\theta}{1+(\cos^2\theta+\sin^2\theta)-2\cos\theta}$

$= \dfrac{2i\sin\theta}{2(1-\cos\theta)} = \dfrac{i\sin\theta}{(1-\cos\theta)} = \dfrac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} \cdot i = \left(\cot\dfrac{\theta}{2}\right)i.$

20. $\dfrac{z_1+z_2+1}{z_1-z_2+i} = \dfrac{(2-i)+(1+i)+1}{(2-i)-(1+i)+i} = \dfrac{4}{(1+i)} \times \dfrac{(1-i)}{(1-i)} = 2(1-i)$

$\Rightarrow \quad \left|\dfrac{z_1+z_2+1}{z_1-z_2+i}\right| = 2 \times \sqrt{1+(-1)^2} = 2\sqrt{2}.$

21. (iv) $(y^2+6-2x)+(y^2-x+1)i = 0 \Rightarrow y^2+6-2x = 0,\ y^2-x+1 = 0$

$\Rightarrow \quad y^2-2x = -6$ and $y^2-x = -1.$

On subtracting, we get $x = 5.$

$\therefore \quad y^2-5 = -1 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2.$

(v) $(x+3i) = (1-i)(2+iy)$

$\Rightarrow \quad (x+3i) = (2+y)+(y-2)i \Rightarrow (x-y-2)+i(5-y) = 0$

$\Rightarrow \quad x-y-2 = 0$ and $5-y = 0.$

(vi) $\{x+(x-2)i\}(3-i)+\{2y+(1-3y)i\}(3+i)=i(3+i)(3-i)$

$\Rightarrow \quad \{3x+(x-2)\}+\{3(x-2)-x\}i+\{6y-(1-3y)\}+\{2y+(3-9y\}i=10i$

$\Rightarrow \quad (4x-2+9y-1)+(2x-6+3-7y)i=10i$

$\Rightarrow \quad (4x+9y-3)+(2x-7y-3)i=10i$

$\Rightarrow \quad 4x+9y=3$ and $2x-7y=13.$

22. $(x-iy)(3+5i)=\overline{(-6-24i)}=(-6+24i)$

$\Rightarrow \quad (3x+5y)+(5x-3y)i=-6+24i$

$\Rightarrow \quad 3x+5y=-6$ and $5x-3y=24.$

23. $(-3+iyx^2)=\overline{(x^2+y+4i)}$

$\Rightarrow \quad (-3+iyx^2)=(x^2+y-4i)$

$\Rightarrow \quad (x^2+y+3)-(4+yx^2)i=0$

$\Rightarrow \quad x^2+y+3=0$ and $4+yx^2=0$

$\Rightarrow \quad x^2+y=-3 \quad \ldots$ (i) and $yx^2=-4 \quad \ldots$ (ii).

Putting $x^2=(-3-y)$ from (i) in (ii), we get

$y(-3-y)=-4 \Rightarrow y^2+3y-4=0 \Rightarrow (y+4)(y-1)=0 \Rightarrow y=-4$ or $y=1.$

Now, $y=1 \Rightarrow x^2=-4 \Rightarrow x$ is imaginary. So, $y\neq 1.$

When $y=-4$, we get $x^2=(-3+4)=1 \Rightarrow x=\pm 1.$

$\therefore \quad (x=1, y=-4)$ or $(x=-1, y=-4).$

25. $(1+i)z=(1-i)\bar{z} \Rightarrow \dfrac{z}{\bar{z}}=\dfrac{(1-i)}{(1+i)}\times\dfrac{(1-i)}{(1-i)}=\dfrac{(1-i)^2}{2}=\dfrac{(1+i^2-2i)}{2}=-i.$

Hence, $z=-i\bar{z}.$

26. Let $z=x+iy$. Then,

$$\frac{z-1}{z+1}=\frac{(x-1)+iy}{(x+1)+iy}\times\frac{(x+1)-iy}{(x+1)-iy}=\frac{(x^2+y^2-1)+(x+y)i}{(x+1)^2+y^2}.$$

Now, $\dfrac{z-1}{z+1}$ is purely imaginary $\Leftrightarrow x^2+y^2-1=0 \Leftrightarrow x^2+y^2=1 \Leftrightarrow |z|=1.$

27. Let $z=(x+iy)$. Then, $z^2=(x^2-y^2)+2ixy.$

Now, $\mathrm{Re}(z^2)=0 \Rightarrow x^2-y^2=0. \quad \ldots$ (i)

$|z|=2 \Rightarrow |z|^2=4 \Rightarrow x^2+y^2=4. \quad \ldots$ (ii)

On solving (i) and (ii), we get

$x^2=2$ and $y^2=2.$

$\therefore \quad x=\pm\sqrt{2}$ and $y=\pm\sqrt{2}.$

$\therefore \quad z=\sqrt{2}(1\pm i)$ or $z=\sqrt{2}(-1\pm i).$

28. Let the required complex number be $z=(x+iy)$. Then,

$|z|=z+1+2i$

$\Rightarrow \quad |x+iy|=(x+iy)+1+2i$

$\Rightarrow \quad \sqrt{x^2+y^2}=(x+1)+(y+2)i$

$\Rightarrow \quad \sqrt{x^2+y^2} = (x+1)$ and $y+2=0$

$\Rightarrow \quad y=-2$ and $\sqrt{x^2+(-2)^2} = (x+1)$

$\Rightarrow \quad y=-2$ and $x^2+4=(x+1)^2 \Rightarrow x=\frac{3}{2}$ and $y=-2$.

SOME RESULTS ON MODULUS AND CONJUGATE OF COMPLEX NUMBERS

THEOREM 1 *For any complex numbers z, z_1 and z_2 prove that:*

(i) $\overline{(\bar{z})} = z$ *(ii)* $z\bar{z} = |z|^2 = \{\text{Re } z\}^2 + \{\text{Im}(z)\}^2$

(iii) $z+\bar{z} = 2\,\text{Re}(z)$ *(iv)* $z-\bar{z} = 2i\cdot\text{Im}(z)$

(v) $z=\bar{z} \Leftrightarrow z$ *is purely real* *(vi)* $z+\bar{z}=0 \Leftrightarrow z$ *is purely imaginary*

(vii) $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$ *(viii)* $\overline{z_1-z_2} = \bar{z}_1-\bar{z}_2$

(ix) $\overline{z_1 z_2} = (\bar{z}_2)(\bar{z}_2)$ *(x)* $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2},\ z_2 \neq 0$

PROOF Let $z=(a+ib)$, $z_1=(a_1+ib_1)$ and $z_2=(a_2+ib_2)$. Then,

(i) $z=(a+ib) \Rightarrow \bar{z} = \overline{(a+ib)} = (a-ib)$

$\Rightarrow \overline{(\bar{z})} = \overline{(a-ib)} = (a+ib) = z.$

Hence, $\overline{(\bar{z})} = z.$

(ii) $z\bar{z} = (a+ib)\overline{(a+ib)} = (a+ib)(a-ib)$

$= (a^2+b^2) = \{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2 = |z|^2.$

$\therefore \quad z\bar{z} = \{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2 = |z|^2.$

(iii) $z+\bar{z} = (a+ib)+\overline{(a+ib)}$

$= (a+ib)+(a-ib) = 2a = 2\text{Re}(z).$

$\therefore \quad z+\bar{z} = 2\,\text{Re}(z).$

(iv) $z-\bar{z} = (a+ib)-\overline{(a+ib)}$

$= (a+ib)-(a-ib) = 2ib = 2i\cdot\text{Im}(z).$

$\therefore \quad z-\bar{z} = 2i\cdot\text{Im}(z).$

(v) $z=\bar{z} \Leftrightarrow (a+ib) = \overline{(a+ib)}$

$\Leftrightarrow (a+ib) = (a-ib) \Leftrightarrow 2ib = 0$

$\Leftrightarrow b=0 \Leftrightarrow \text{Im}(z) = 0.$

$\therefore \quad z=\bar{z} \Leftrightarrow \text{Im}(z)=0 \Leftrightarrow z$ is purely real.

(vi) $z+\bar{z}=0 \Leftrightarrow (a+ib)+\overline{(a+ib)} = 0$

$\Leftrightarrow (a+ib)+(a-ib) = 0 \Leftrightarrow 2a=0 \Leftrightarrow a=0$

$\Leftrightarrow z$ is purely imaginary.

$\therefore \quad z+\bar{z}=0 \Leftrightarrow z$ is purely imaginary.

(vii) $\overline{z_1 + z_2} = \overline{(a_1 + ib_1) + (a_2 + ib_2)}$

$= \overline{(a_1 + a_2) + i(b_1 + b_2)} = (a_1 + a_2) - i(b_1 + b_2)$

$= (a_1 - ib_1) + (a_2 - ib_2) = \bar{z}_1 + \bar{z}_2.$

$\therefore \quad \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2.$

(viii) $\overline{z_1 - z_2} = \overline{(a_1 + ib_1) - (a_2 + ib_2)}$

$= \overline{(a_1 - a_2) + i(b_1 - b_2)} = (a_1 - a_2) - i(b_1 - b_2)$

$= (a_1 - ib_1) - (a_2 - ib_2) = \bar{z}_1 - \bar{z}_2.$

$\therefore \quad \overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2.$

(ix) $z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)$

$\Rightarrow \quad \overline{z_1 z_2} = \overline{(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)}$

$= (a_1 a_2 - b_1 b_2) - i(a_1 b_2 + b_1 a_2)$

$= (a_1 - ib_1)(a_2 - ib_2) = \bar{z}_1 \, \bar{z}_2.$

$\therefore \quad \overline{z_1 z_2} = \bar{z}_1 \, \bar{z}_2.$

(x) $\dfrac{z_1}{z_2} = \dfrac{(a_1 + ib_1)}{(a_2 + ib_2)} = \dfrac{(a_1 + ib_1)}{(a_2 + ib_2)} \times \dfrac{(a_2 - ib_2)}{(a_2 - ib_2)}$

$= \dfrac{(a_1 + ib_1)(a_2 - ib_2)}{(a_2^2 + b_2^2)} = \dfrac{(a_1 a_2 + b_1 b_2) + i(b_1 a_2 - a_1 b_2)}{(a_2^2 + b_2^2)}$

$= \dfrac{(a_1 a_2 + b_1 b_2)}{(a_2^2 + b_2^2)} + i\dfrac{(b_1 a_2 - a_1 b_2)}{(a_2^2 + b_2^2)}$

$\therefore \quad \overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{(a_1 a_2 + b_1 b_2)}{(a_2^2 + b_2^2)} - i\dfrac{(b_1 a_2 - a_1 b_2)}{(a_2^2 + b_2^2)}.$ ... (i)

Also, $\dfrac{\bar{z}_1}{\bar{z}_2} = \bar{z}_1 \cdot \dfrac{1}{\bar{z}_2} = (a_1 - ib_1) \cdot \left\{\dfrac{1}{(a_2 - ib_2)} \times \dfrac{(a_2 + ib_2)}{(a_2 + ib_2)}\right\}$

$= \dfrac{(a_1 - ib_1)(a_2 + ib_2)}{(a_2^2 + b_2^2)} = \dfrac{(a_1 a_2 + b_1 b_2)}{(a_2^2 + b_2^2)} - i\dfrac{(b_1 a_2 - a_1 b_2)}{(a_2^2 + b_2^2)}.$ ... (ii)

From (i) and (ii), we get

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}.$$

THEOREM 2 *For all complex numbers z, z_1 and z_2, prove that:*

(i) $z\bar{z} = |z|^2$ *(ii)* $|z| = |\bar{z}| = |-z|$

(iii) $|z| = 0 \Leftrightarrow z = 0$ *(iv)* $-|z| \le \mathrm{Re}(z) \le |z|$

(v) $-|z| \le \mathrm{Im}(z) \le |z|$ *(vi)* $|z_1 z_2| = |z_1||z_2|$

(vii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}, z_2 \neq 0$

PROOF Let $z = (a + ib)$, $z_1 = (a_1 + ib_1)$ and $z_2 = (a_2 + ib_2)$. Then,

(i) $z\bar{z} = (a+ib)\overline{(a+ib)} = (a+ib)(a-ib) = (a^2+b^2) = |z|^2$.

$\therefore \quad z\,\bar{z} = |z|^2$.

(ii) $|z| = |a+ib| = \sqrt{a^2+b^2}$, $|\bar{z}| = |a-ib| = \sqrt{a^2+(-b)^2} = \sqrt{a^2+b^2}$

and $|-z| = |-(a+ib)| = |(-a)+i(-b)| = \sqrt{(-a)^2+(-b)^2} = \sqrt{a^2+b^2}$.

$\therefore \quad |z| = |\bar{z}| = |-z|$.

(iii) $|z| = 0 \Leftrightarrow |z|^2 = 0 \Leftrightarrow |a+ib|^2 = 0$

$\Leftrightarrow a^2+b^2 = 0 \Leftrightarrow a = 0$ and $b = 0 \Leftrightarrow z = 0$.

(iv) Let $z = (a+ib)$. Then, $|z| = \sqrt{a^2+b^2}$.

Clearly, $-\sqrt{a^2+b^2} \le a \le \sqrt{a^2+b^2}$.

$\therefore \quad -|z| \le \mathrm{Re}(z) \le |z|$.

(v) Let $z = (a+ib)$. Then, $|z| = \sqrt{a^2+b^2}$.

Clearly, $-\sqrt{a^2+b^2} \le b \le \sqrt{a^2+b^2}$.

$\therefore \quad -|z| \le \mathrm{Im}(z) \le |z|$.

(vi) $z_1 z_2 = (a_1+ib_1)(a_2+ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2)$.

$$\therefore \quad |z_1z_2| = \sqrt{(a_1a_2-b_1b_2)^2+(a_1b_2+b_1a_2)^2}$$
$$= \sqrt{a_1^2a_2^2+b_1^2b_2^2+a_1^2b_2^2+b_1^2a_2^2}$$
$$= \sqrt{a_1^2(a_2^2+b_2^2)+b_1^2(a_2^2+b_2^2)} = \sqrt{(a_1^2+b_1^2)(a_2^2+b_2^2)}$$
$$= \left\{\sqrt{a_1^2+b_1^2}\right\}\left\{\sqrt{a_2^2+b_2^2}\right\} = |z_1|\cdot|z_2|$$

$\therefore \quad |z_1z_2| = |z_1|\cdot|z_2|$.

(vii) $\dfrac{z_1}{z_2} = z_1 \cdot \dfrac{1}{z_2} = (a_1+ib_1)\cdot\left\{\dfrac{1}{(a_2+ib_2)} \times \dfrac{(a_2-ib_2)}{(a_2-ib_2)}\right\}$

$$= \frac{(a_1+ib_1)(a_2-ib_2)}{(a_2+ib_2)(a_2-ib_2)} = \frac{(a_1a_2+b_1b_2)+i(b_1a_2-a_1b_2)}{(a_2^2+b_2^2)}$$
$$= \frac{(a_1a_2+b_1b_2)}{(a_2^2+b_2^2)} + i\frac{(a_2b_1-a_1b_2)}{(a_2^2+b_2^2)}$$

$$\Rightarrow \quad \left|\frac{z_1}{z_2}\right| = \sqrt{\left\{\frac{a_1a_2+b_1b_2}{a_2^2+b_2^2}\right\}^2 + \left\{\frac{a_2b_1-a_1b_2}{a_2^2+b_2^2}\right\}^2}$$
$$= \sqrt{\frac{(a_1a_2+b_1b_2)^2+(a_2b_1-a_1b_2)^2}{(a_2^2+b_2^2)^2}}$$
$$= \sqrt{\frac{a_1^2a_2^2+b_1^2b_2^2+a_2^2b_1^2+a_1^2b_2^2}{(a_2^2+b_2^2)^2}}$$

$$= \sqrt{\frac{a_1^2(a_2^2+b_2^2)+b_1^2(a_2^2+b_2^2)}{(a_2^2+b_2^2)^2}}$$

$$= \sqrt{\frac{(a_2^2+b_2^2)(a_1^2+b_1^2)}{(a_2^2+b_2^2)^2}} = \sqrt{\frac{(a_1^2+b_1^2)}{(a_2^2+b_2^2)}} = \frac{\sqrt{a_1^2+b_1^2}}{\sqrt{a_2^2+b_2^2}} = \frac{|z_1|}{|z_2|}.$$

Hence, $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \neq 0$.

THEOREM 3 *For all complex numbers z_1 and z_2, prove that:*

(i) $\text{Re}(z_1 z_2) = \text{Re}(z_1)\cdot\text{Re}(z_2) - \text{Im}(z_1)\cdot\text{Im}(z_2)$

(ii) $\text{Im}(z_1 z_2) = \text{Re}(z_1)\cdot\text{Im}(z_2) + \text{Im}(z_1)\cdot\text{Re}(z_2)$

PROOF Let $z_1 = (a_1 + ib_1)$ *and* $z_2 = (a_2 + ib_2)$. Then,

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2)$$

$$\Rightarrow \quad z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2).$$

(i) $\text{Re}(z_1 z_2) = (a_1 a_2 - b_1 b_2)$

$$= \text{Re}(z_1)\cdot\text{Re}(z_2) - \text{Im}(z_1)\cdot\text{Im}(z_2).$$

(ii) $\text{Im}(z_1 z_2) = (a_1 b_2 + b_1 a_2)$

$$= \text{Re}(z_1)\cdot\text{Im}(z_2) + \text{Im}(z_1)\cdot\text{Re}(z_2).$$

THEOREM 4 *For all $a, b \in R$ and $z_1, z_2 \in C$, prove that*

$$|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2).$$

PROOF We have

$$|az_1 - bz_2|^2 = (az_1 - bz_2)\overline{(az_1 - bz_2)}$$

$$= (az_1 - bz_2)(a\bar{z}_1 - b\bar{z}_2) \qquad [\because\ a, b \in R]$$

$$= a^2 z_1\bar{z}_1 + b^2 z_2\bar{z}_2 - abz_1\bar{z}_2 - abz_2\bar{z}_1$$

$$= a^2|z_1|^2 + b^2|z_2|^2 - ab(z_1\bar{z}_2 + z_2\bar{z}_1). \qquad \ldots \text{(i)}$$

$$|bz_1 + az_2|^2 = (bz_1 + az_2)\overline{(bz_1 + az_2)}$$

$$= (bz_1 + az_2)(b\bar{z}_1 + a\bar{z}_2) \qquad [\because\ a, b \in R]$$

$$= b^2 z_1\bar{z}_1 + a^2 z_2\bar{z}_2 + abz_1\bar{z}_2 + abz_2\bar{z}_1$$

$$= b^2|z_1|^2 + a^2|z_2|^2 + ab(z_1\bar{z}_2 + z_2\bar{z}_1). \qquad \ldots \text{(ii)}$$

Adding the corresponding sides of (i) and (ii), we get

$$|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2).$$

SOLVED EXAMPLES

EXAMPLE 1 *If* $\left|\frac{z-5i}{z+5i}\right| = 1$, *show that z is a real number.*

SOLUTION Let $z = (x + iy)$. Then,

$$\left|\frac{z-5i}{z+5i}\right| = 1 \ \Rightarrow\ \frac{|z-5i|}{|z+5i|} = 1 \qquad \left[\because\ \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}\right]$$

$\Rightarrow \quad |z-5i| = |z+5i| \Rightarrow |z-5i|^2 = |z+5i|^2$

$\Rightarrow \quad |(x+iy)-5i|^2 = |(x+iy)+5i|^2 \qquad [\because\ z = (x+iy)]$

$\Rightarrow \quad |x+i(y-5)|^2 = |x+i(y+5)|^2$

$\Rightarrow \quad x^2+(y-5)^2 = x^2+(y+5)^2 \qquad [\because\ |x+iy|^2 = (x^2+y^2)]$

$\Rightarrow \quad (y+5)^2-(y-5)^2 = 0 \Rightarrow 4\times y\times 5 = 0 \Rightarrow y = 0.$

$\therefore \quad z = x+i0 \Rightarrow z = x,$ where x is real.

Hence, z is a real number.

EXAMPLE 2 *If* $(a+ib) = \dfrac{(x+i)^2}{(2x^2+1)}$ *then prove that* $(a^2+b^2) = \dfrac{(x^2+1)^2}{(2x^2+1)^2}$.

SOLUTION We have

$$(a+ib) = \frac{(x+i)^2}{(2x^2+1)} = \frac{(x^2+i^2+2ix)}{(2x^2+1)} = \frac{(x^2-1)+i(2x)}{(2x^2+1)}$$

$$\Rightarrow \quad (a+ib) = \frac{(x^2-1)}{(2x^2+1)} + i\cdot\frac{2x}{(2x^2+1)}$$

$$\Rightarrow \quad |a+ib|^2 = \left|\frac{(x^2-1)}{(2x^2+1)} + i\cdot\frac{2x}{(2x^2+1)}\right|^2$$

$$\Rightarrow \quad (a^2+b^2) = \left\{\frac{(x^2-1)^2}{(2x^2+1)^2} + \frac{4x^2}{(2x^2+1)^2}\right\}$$

$$= \frac{(x^2-1)^2+4x^2}{(2x^2+1)^2} = \frac{(x^2+1)^2}{(2x^2+1)^2}.$$

Hence, $(a^2+b^2) = \dfrac{(x^2+1)^2}{(2x^2+1)^2}$.

EXAMPLE 3 *If* $(p+iq) = \dfrac{(a+i)^2}{(2a-i)}$ *then prove that* $(p^2+q^2) = \dfrac{(a^2+1)^2}{(4a^2+1)}$.

SOLUTION We have

$$(p+iq) = \frac{(a+i)^2}{(2a-i)} = \frac{(a^2+i^2+2ai)}{(2a-i)} = \frac{(a^2-1)+2ai}{(2a-i)}$$

$$\Rightarrow \quad |p+iq|^2 = \frac{|(a^2-1)+2ai|^2}{|2a-i|^2} = \frac{(a^2-1)^2+4a^2}{\{4a^2+(-1)^2\}} = \frac{(a^2+1)^2}{(4a^2+1)}$$

$$\Rightarrow \quad (p^2+q^2) = \frac{(a^2+1)^2}{(4a^2+1)}.$$

Hence, $(p^2+q^2) = \dfrac{(a^2+1)^2}{(4a^2+1)}$.

EXAMPLE 4 *Let z be a complex number such that $\left|\frac{1-iz}{z-i}\right|=1$. Show that z is purely real.*

SOLUTION Let $z = x+iy$. Then,

$$\left|\frac{1-iz}{z-i}\right|=1 \Rightarrow \frac{|1-iz|}{|z-i|}=1 \qquad \left[\because \left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}\right]$$

$$\Rightarrow \quad |1-iz| = |z-i| \Rightarrow |1-iz|^2 = |z-i|^2$$

$$\Rightarrow \quad |1-i(x+iy)|^2 = |(x+iy)-i|^2$$

$$\Rightarrow \quad |(1+y)-ix|^2 = |x+(y-1)i|^2$$

$$\Rightarrow \quad (1+y)^2+(-x)^2 = x^2+(y-1)^2$$

$$\Rightarrow \quad 1+y^2+2y+x^2 = x^2+y^2-2y+1$$

$$\Rightarrow \quad 4y=0 \Rightarrow y=0.$$

$\therefore$ $z = x+0y$ and hence z is purely real.

EXAMPLE 5 *If z_1 and z_2 are complex numbers such that $\frac{2z_1}{3z_2}$ is purely imaginary then prove that $\left|\frac{z_1-z_2}{z_1+z_2}\right|=1$.*

SOLUTION Let $\frac{2z_1}{3z_2}=ki$ for some real number k. Then,

$$\frac{2z_1}{3z_2}=ki \Rightarrow \frac{z_1}{z_2}=\frac{3ki}{2}. \qquad \ldots \text{(i)}$$

$$\therefore \quad \left|\frac{z_1-z_2}{z_1+z_2}\right| = \frac{|z_1-z_2|}{|z_1+z_2|} = \frac{\left|\frac{z_1}{z_2}-1\right|}{\left|\frac{z_1}{z_2}+1\right|} \qquad \left[\because \left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}\right]$$

$$= \frac{\left|\frac{3ki}{2}-1\right|}{\left|\frac{3ki}{2}+1\right|} = \frac{|3ki-2|}{|3ki+2|} = \frac{\sqrt{9k^2+4}}{\sqrt{9k^2+4}} = 1 \qquad \text{[using (i)].}$$

Hence, $\left|\frac{z_1-z_2}{z_1+z_2}\right|=1$.

EXAMPLE 6 *If $|z_1| = |z_2| = |z_3| = \ldots = |z_n| = 1$ then prove that*

$$\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}+\ldots+\frac{1}{z_n}\right| = |z_1+z_2+z_3+\ldots+z_n|.$$

SOLUTION We have

$$|z_1| = |z_2| = |z_3| = \ldots = |z_n| = 1$$

$$\Rightarrow \quad |z_1|^2 = |z_2|^2 = |z_3|^2 = \ldots = |z_n|^2 = 1$$

$$\Rightarrow \quad z_1\bar{z}_1 = 1, z_2\bar{z}_2 = 1, z_3\bar{z}_3 = 1, \ldots, z_n\bar{z}_n = 1$$

$$\Rightarrow \quad \frac{1}{z_1}=\bar{z}_1, \frac{1}{z_2}=\bar{z}_2, \frac{1}{z_3}=\bar{z}_3, \ldots, \frac{1}{z_n}=\bar{z}_n$$

$$\Rightarrow \quad \left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}+\ldots+\frac{1}{z_n}\right| = |\bar{z}_1+\bar{z}_2+\bar{z}_3+\ldots+\bar{z}_n|$$

$$= |\overline{z_1+z_2+z_3+\ldots+z_n}|$$

$$= |z_1+z_2+z_3+\ldots+z_n| \qquad [\because\ |\bar{z}| = |z|\ ].$$

$$\therefore \quad \left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}+\ldots+\frac{1}{z_n}\right| = |z_1+z_2+z_3+\ldots+z_n|.$$

EXAMPLE 7 *If* $\frac{a+ib}{c+id} = x+iy$, *prove that*

(i) $\frac{a-ib}{c-id} = x-iy$ *and* *(ii)* $\frac{a^2+b^2}{c^2+d^2} = x^2+y^2.$

SOLUTION (i) $\left(\frac{a+ib}{c+id}\right) = (x+iy) \Rightarrow \overline{\left(\frac{a+ib}{c+id}\right)} = \overline{(x+iy)}$

$$\Rightarrow \frac{\overline{(a+ib)}}{\overline{(c+id)}} = (x-iy) \Rightarrow \frac{(a-ib)}{(c-id)} = (x-iy).$$

(ii) We have

$$\frac{(a+ib)}{(c+id)} = (x+iy) \text{ and } \frac{(a-ib)}{(c-id)} = (x-iy)$$

$$\Rightarrow \quad \frac{(a+ib)}{(c+id)} \times \frac{(a-ib)}{(c-id)} = (x+iy)(x-iy)$$

$$\Rightarrow \quad \frac{(a+ib)(a-ib)}{(c+id)(c-id)} = (x+iy)(x-iy)$$

$$\Rightarrow \quad \frac{(a^2+b^2)}{(c^2+d^2)} = (x^2+y^2).$$

EXAMPLE 8 *If* $(x+iy) = \sqrt{\frac{a+ib}{c+id}}$, *prove that* $(x^2+y^2)^2 = \frac{a^2+b^2}{c^2+d^2}$.

SOLUTION We have

$$(x+iy) = \sqrt{\frac{a+ib}{c+id}} = \frac{\sqrt{a+ib}}{\sqrt{c+id}}$$

$$\Rightarrow \quad (x-iy) = \frac{\sqrt{a-ib}}{\sqrt{c-id}}$$

$$\Rightarrow \quad (x+iy)(x-iy) = \frac{\sqrt{a+ib}}{\sqrt{c+id}} \times \frac{\sqrt{a-ib}}{\sqrt{c-id}} = \frac{\sqrt{(a+ib)(a-ib)}}{\sqrt{(c+id)(c-id)}}$$

$$\Rightarrow \quad (x^2+y^2) = \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}} \Rightarrow (x^2+y^2)^2 = \frac{(a^2+b^2)}{(c^2+d^2)}.$$

Hence, $(x^2+y^2)^2 = \frac{(a^2+b^2)}{(c^2+d^2)}$.

EXAMPLE 9 *If* $(1+i)(1+2i)(1+3i)\ldots(1+ni) = (x+iy)$ *then prove that*

$$2\times 5\times 10\times\ldots\times(1+n^2) = (x^2+y^2).$$

SOLUTION $(1+i)(1+2i)(1+3i)\ldots(1+ni) = (x+iy)$

$\Rightarrow \quad |(1+i)(1+2i)(1+3i)\ldots(1+ni)|^2 = |x+iy|^2$

$\Rightarrow \quad |1+i|^2\times|1+2i|^2\times|1+3i|^2\times\ldots\times|1+ni|^2 = (x^2+y^2)$

$\Rightarrow \quad (\sqrt{2})^2\times(\sqrt{5})^2\times(\sqrt{10})^2\times\ldots\times(\sqrt{1+n^2})^2 = (x^2+y^2)$

$\Rightarrow \quad 2\times 5\times 10\times\ldots\times(1+n^2) = (x^2+y^2).$

Hence, $2\times 5\times 10\times\ldots\times(1+n^2) = (x^2+y^2).$

EXAMPLE 10 *If* $(a+ib)(c+id)(e+if)(g+ih) = (A+iB)$ *then show that*

$$(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = (A^2+B^2).$$

SOLUTION $(a+ib)(c+id)(e+if)(g+ih) = (A+iB)$

$\Rightarrow \quad |(a+ib)(c+id)(e+if)(g+ih)| = |A+iB|$

$\Rightarrow \quad |a+ib|\cdot|c+id|\cdot|e+if|\cdot|g+ih| = |A+iB|$

$\Rightarrow \quad |a+ib|^2\cdot|c+id|^2\cdot|e+if|^2\cdot|g+ih|^2 = |A+iB|^2$

$\Rightarrow \quad (a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = (A^2+B^2).$

Hence, $(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = (A^2+B^2).$

EXAMPLE 11 *If* α *and* β *are different complex numbers such that* $|\beta| = 1$, *show that* $\left|\dfrac{\beta-\alpha}{1-\bar{\alpha}\beta}\right| = 1.$

SOLUTION We have

$$\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|^2 = \left\{\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right\}\overline{\left\{\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right\}} \qquad [\because |z|^2 = z\bar{z}]$$

$$= \frac{(\beta-\alpha)}{(1-\bar{\alpha}\beta)}\cdot\frac{(\bar{\beta}-\bar{\alpha})}{(1-\alpha\bar{\beta})} = \frac{(\beta-\alpha)(\bar{\beta}-\bar{\alpha})}{(1-\bar{\alpha}\beta)(1-\alpha\bar{\beta})}$$

$$= \frac{\beta\bar{\beta}+\alpha\bar{\alpha}-\beta\bar{\alpha}-\alpha\bar{\beta}}{1-\alpha\bar{\beta}-\bar{\alpha}\beta+(\alpha\bar{\alpha})(\beta\bar{\beta})}$$

$$= \frac{|\beta|^2+|\alpha|^2-(\alpha\bar{\beta}+\beta\bar{\alpha})}{1-(\alpha\bar{\beta}+\beta\bar{\alpha})+|\alpha|^2\,|\beta|^2} \qquad [\because \alpha\bar{\alpha} = |\alpha|^2, \beta\bar{\beta} = |\beta|^2]$$

$$= \frac{1+|\alpha|^2-(\alpha\bar{\beta}+\beta\bar{\alpha})}{1+|\alpha|^2-(\alpha\bar{\beta}+\beta\bar{\alpha})} = 1 \qquad [\because |\beta| = 1].$$

Hence, $\left|\dfrac{\beta-\alpha}{1-\bar{\alpha}\beta}\right| = 1.$

EXAMPLE 12 *If* $|z^2-1| = |z|^2+1$ *then show that z is imaginary.*

SOLUTION Let $z = (x+iy)$. Then, $z^2 = (x^2-y^2)+2ixy$ and $|z|^2 = (x^2+y^2)$.

Now, $|z^2-1| = |z|^2+1$

$\Rightarrow \quad |(x^2-y^2-1)+i(2xy)| = (x^2+y^2+1)$

$\Rightarrow \quad |(x^2-y^2-1)+i(2xy)|^2 = (x^2+y^2+1)^2$

$\Rightarrow \quad (x^2-y^2-1)^2+4x^2y^2 = (x^2+y^2+1)^2$

$\Rightarrow \quad [x^2+(y^2+1)]^2-[x^2-(y^2+1)]^2 = 4x^2y^2$

$\Rightarrow \quad 4x^2(y^2+1) = 4x^2y^2 \Rightarrow 4x^2\{(y^2+1)-y^2\} = 0$

$\Rightarrow \quad 4x^2 = 0 \Rightarrow x = 0.$

Hence, $z = 0+iy$, which shows that *z* is imaginary.

EXAMPLE 13 *If* $iz^3+z^2-z+i = 0$ *then show that* $|z| = 1$.

SOLUTION We have

$iz^3+z^2-z+i = 0$

$\Rightarrow \quad z^3-iz^2+iz+1 = 0$ [on dividing both sides by *i*]

$\Rightarrow \quad z^2(z-i)+i(z-i) = 0$

$\Rightarrow \quad (z-i)(z^2+i) = 0$

$\Rightarrow \quad z = i \text{ or } z^2 = -i.$

Now, $z = i \Rightarrow |z| = |i| \Rightarrow |z| = 1.$

And, $z^2 = -i \Rightarrow |z^2| = |-i| = 1 \Rightarrow |z|^2 = 1 \Rightarrow |z| = 1.$

Hence, in either case, we have $|z| = 1$.

EXAMPLE 14 *Find all nonzero complex numbers z satisfying* $\bar{z} = iz^2$.

SOLUTION Let $z = x+iy$. Then, $\bar{z} = x-iy$ and $z^2 = (x^2-y^2)+2ixy$.

$\therefore \quad \bar{z} = iz^2 \Rightarrow x-iy = i(x^2-y^2)-2xy$

$\Rightarrow (x+2xy)-i(x^2-y^2+y) = 0. \quad \ldots$ (i)

On equating real parts and imaginary parts on both sides of (i) separately, we get

$x+2xy = 0 \quad \ldots$ (ii) and $x^2-y^2+y = 0 \quad \ldots$ (iii).

From (ii), we get

$x+2xy = 0 \Rightarrow x(1+2y) = 0$

$\Rightarrow x = 0 \text{ or } 1+2y = 0 \Rightarrow x = 0 \text{ or } y = \frac{-1}{2}.$

Case I *When* $x = 0$.

Putting $x = 0$ in (iii), we get

$-y^2+y = 0 \Rightarrow -y(y-1) = 0 \Rightarrow y = 0 \text{ or } y = 1.$

Thus, $(x = 0, y = 0)$ or $(x = 0, y = 1)$

$\therefore \quad z = (0 + i0)$ or $z = (0 + 1 \cdot i)$.

Case II *When* $y = \frac{-1}{2}$.

Putting $y = \frac{-1}{2}$ in (iii), we get

$$x^2 - \left(\frac{-1}{2}\right)^2 + \left(\frac{-1}{2}\right) = 0 \Rightarrow x^2 - \frac{1}{4} - \frac{1}{2} = 0$$

$$\Rightarrow \quad x^2 = \left(\frac{1}{4} + \frac{1}{2}\right) = \frac{3}{4} \Rightarrow x = \pm\frac{\sqrt{3}}{2}.$$

$$\therefore \quad \left(x = \frac{\sqrt{3}}{2}, y = \frac{-1}{2}\right) \text{ or } \left(x = \frac{-\sqrt{3}}{2}, y = \frac{-1}{2}\right)$$

$$\therefore \quad z = \left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) \text{ or } z = \left(\frac{-\sqrt{3}}{2} - \frac{1}{2}i\right).$$

Hence, the required nonzero complex numbers are

$$i, \left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) \text{and} \left(\frac{-\sqrt{3}}{2} - \frac{1}{2}i\right).$$

EXAMPLE 15 *Solve the equation,* $z^2 = \bar{z}$, *where z is a complex number.*

SOLUTION Let $z = x + iy$. Then,

$$z^2 = \bar{z} \Rightarrow (x + iy)^2 = x - iy$$

$$\Rightarrow (x^2 - y^2) + 2ixy = x - iy. \qquad \ldots \text{(i)}$$

Equating real parts and imaginary parts on both sides of (i) separately, we get

$$x^2 - y^2 = x \qquad \ldots \text{(ii)} \quad \text{and} \quad 2xy = -y \qquad \ldots \text{(iii)}.$$

From (iii), we get

$$2xy + y = 0 \Rightarrow y(2x + 1) = 0 \Rightarrow y = 0 \text{ or } x = \frac{-1}{2}.$$

Case I *When* $y = 0$.

Putting $y = 0$ in (ii), we get

$$x^2 - x = 0 \Rightarrow x(x - 1) = 0 \Rightarrow x = 0 \text{ or } x = 1.$$

$\therefore \quad (x = 0, y = 0)$ or $(x = 1, y = 0)$

Thus, $z = (0 + i0)$ or $z = (1 + i0)$.

Case II *When* $x = \frac{-1}{2}$.

Putting $x = \frac{-1}{2}$ in (ii), we get

$$\left(\frac{-1}{2}\right)^2 - y^2 = \left(\frac{-1}{2}\right) \Rightarrow y^2 = \left(\frac{1}{4} + \frac{1}{2}\right) = \frac{3}{4} \Rightarrow y = \pm\frac{\sqrt{3}}{2}.$$

$$\therefore \quad \left(x=\frac{-1}{2}, y=\frac{\sqrt{3}}{2}\right) \text{ or } \left(x=\frac{-1}{2}, y=\frac{-\sqrt{3}}{2}\right).$$

$$\text{Thus, } z=\left(\frac{-1}{2}+\frac{\sqrt{3}}{2}i\right) \text{ or } z=\left(\frac{-1}{2}-\frac{\sqrt{3}}{2}i\right).$$

Hence, $z=0, 1, \left(\frac{-1}{2}+\frac{\sqrt{3}}{2}i\right)$ and $\left(\frac{-1}{2}-\frac{\sqrt{3}}{2}i\right)$ are the required roots of the given equation.

EXAMPLE 16 *For all complex numbers z_1 and z_2, prove that*

$$|z_1+z_2|^2+|z_1-z_2|^2=2(|z_1|^2+|z_2|^2).$$

SOLUTION We have

$$|z_1+z_2|^2=(z_1+z_2)(\overline{z_1+z_2})$$

$$=(z_1+z_2)(\bar{z}_1+\bar{z}_2) \qquad [\because \overline{(z_1+z_2)}=\bar{z}_1+\bar{z}_2]$$

$$=z_1\bar{z}_1+z_2\bar{z}_2+z_1\bar{z}_2+z_2\bar{z}_1$$

$$=|z_1|^2+|z_2|^2+z_1\bar{z}_2+z_2\bar{z}_1. \qquad \ldots \text{(i)}$$

$$|z_1-z_2|^2=(z_1-z_2)(\overline{z_1-z_2})=(z_1-z_2)(\bar{z}_1-\bar{z}_2)$$

$$=z_1\bar{z}_1+z_2\bar{z}_2-z_1\bar{z}_2-z_2\bar{z}_1$$

$$=|z_1|^2+|z_2|^2-z_1\bar{z}_2-z_2\bar{z}_1. \qquad \ldots \text{(ii)}$$

On adding the corresponding sides of (i) and (ii), we get

$$|z_1+z_2|^2+|z_1-z_2|^2=2(|z_1|^2+|z_2|^2).$$

EXERCISE 5C

1. Express each of the following in the form $(a+ib)$ and find its conjugate:

(i) $\frac{1}{(4+3i)}$ (ii) $(2+3i)^2$ (iii) $\frac{(2-i)}{(1-2i)^2}$

(iv) $\frac{(1+i)(1+2i)}{(1+3i)}$ (v) $\left(\frac{1+2i}{2+i}\right)^2$ (vi) $\frac{(2+i)}{(3-i)(1+2i)}$

2. Express each of the following in the form $(a+ib)$ and find its multiplicative inverse:

(i) $\frac{1+2i}{1-3i}$ (ii) $\frac{(1+7i)}{(2-i)^2}$ (iii) $\frac{-4}{(1+i\sqrt{3})}$

3. If $(x+iy)^3=(u+iv)$ then prove that $\left(\frac{u}{x}+\frac{v}{y}\right)=4(x^2-y^2)$.

4. If $(x+iy)^{1/3}=(a+ib)$ then prove that $\left(\frac{x}{a}+\frac{y}{b}\right)=4(a^2-b^2)$.

5. Express $(1-2i)^{-3}$ in the form $(a+ib)$.

6. Find real values of x and y for which

$$(x^4 + 2xi) - (3x^2 + iy) = (3 - 5i) + (1 + 2iy).$$

7. If $z^2 + |z|^2 = 0$, show that z is purely imaginary.

8. If $\frac{z-1}{z+1}$ is purely imaginary and $z \neq -1$, show that $|z| = 1$.

9. If z_1 is a complex number other than –1 such that $|z_1| = 1$ and $z_2 = \frac{z_1 - 1}{z_1 + 1}$ then show that z_2 is purely imaginary.

10. For all $z \in C$, prove that

(i) $\frac{1}{2}(z + \bar{z}) = \text{Re}(z)$ (ii) $\frac{1}{2i}(z - \bar{z}) = \text{Im}(z)$ (iii) $z\bar{z} = |z|^2$

(iv) $(z + \bar{z})$ is real (v) $(z - \bar{z})$ is 0 or imaginary.

11. If $z_1 = (1 + i)$ and $z_2 = (-2 + 4i)$, prove that $\text{Im}\left(\frac{z_1 z_2}{\bar{z}_1}\right) = 2$.

12. If a and b are real numbers such that $a^2 + b^2 = 1$ then show that a real value of x will satisfy the equation, $\frac{1 - ix}{1 + ix} = (a - ib)$.

ANSWERS (EXERCISE 5C)

1. (i) $z = \left(\frac{4}{25} - \frac{3}{25}i\right), \bar{z} = \left(\frac{4}{25} + \frac{3}{25}i\right)$ (ii) $z = (-5 + 12i), \bar{z} = (-5 - 12i)$

(iii) $z = \left(\frac{-2}{25} + \frac{11}{25}i\right), \bar{z} = \left(\frac{-2}{25} - \frac{11}{25}i\right)$ (iv) $z = \left(\frac{4}{5} + \frac{3}{5}i\right), \bar{z} = \left(\frac{4}{5} - \frac{3}{5}i\right)$

(v) $z = \left(\frac{7}{25} + \frac{24}{25}i\right), \bar{z} = \left(\frac{7}{25} - \frac{24}{25}i\right)$ (vi) $z = \left(\frac{3}{10} - \frac{1}{10}i\right), \bar{z} = \left(\frac{3}{10} + \frac{1}{10}i\right)$

2. (i) $z = \left(\frac{-1}{2} + \frac{1}{2}i\right), z^{-1} = (-1 - i)$ (ii) $z = (-1 + i), z^{-1} = \left(\frac{-1}{2} - \frac{1}{2}i\right)$

(iii) $z = (-1 + i\sqrt{3}), z^{-1} = \left(\frac{-1}{4} - \frac{\sqrt{3}}{4}i\right)$

5. $\left(\frac{-11}{125} - \frac{2}{125}i\right)$ **6.** $(x = 2, y = 3)$ or $\left(x = -2, y = \frac{1}{3}\right)$

HINTS TO SOME SELECTED QUESTIONS

3. $(u + iv) = (x + iy)^3 = x^3 - iy^3 + 3ixy(x + iy)$

$\Rightarrow \quad (u + iv) = (x^3 - 3xy^2) + i(3x^2y - y^3)$

$\Rightarrow \quad u = x^3 - 3x^2y$ and $v = 3x^2y - y^3$

$\Rightarrow \quad \left(\frac{u}{x} + \frac{v}{y}\right) = (x^2 - 3y^2) + (3x^2 - y^2) = 4(x^2 - y^2).$

4. $(x + iy) = (a + ib)^3 = a^3 - ib^3 + 3iab(a + ib)$

$\Rightarrow \quad (x + iy) = (a^3 - 3ab^2) + i(3a^2b - b^3)$

$\Rightarrow \quad x = a^3 - 3ab^2$ and $y = 3a^2b - b^3$

$\Rightarrow \quad \left(\frac{x}{a} + \frac{y}{b}\right) = (a^2 - 3b^2) + (3a^2 - b^2) = 4(a^2 - b^2).$

5. $(1-2i)^{-3} = \frac{1}{(1-2i)^3} = \frac{1}{\{1 - 8i^3 - 6i(1-2i)\}}$

$= \frac{1}{(-11+2i)} \times \frac{(-11-2i)}{(-11-2i)} = \frac{(-11-2i)}{125}.$

6. $(x^4 - 3x^2) + i(2x - y) = 4 + (2y - 5)i$

$\Rightarrow \quad (x^4 - 3x^2 - 4) + i(2x - y - 2y + 5) = 0$

$\Rightarrow \quad x^4 - 3x^2 - 4 = 0$ and $2x - 3y + 5 = 0.$

Now, $x^4 - 3x^2 - 4 = 0 \Rightarrow (x^2 - 4)(x^2 + 1) = 0 \Rightarrow x = 2$ or $x = -2.$

$(x = 2 \Rightarrow y = 3)$ and $\left(x = -2 \Rightarrow y = \frac{1}{3}\right).$

7. Let $z = (x + iy)$. Then,

$z^2 + |z|^2 = 0 \Rightarrow x^2 - y^2 + 2ixy + x^2 + y^2 = 0$

$\Rightarrow x^2 + ixy = 0 \Rightarrow x^2 = 0$ and $xy = 0 \Rightarrow x = 0.$

$\therefore \quad z$ is purely imaginary.

8. Let $z = (x + iy)$. Then,

$$\frac{z-1}{z+1} = \frac{(x+iy)-1}{(x+iy)+1} = \frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy} = \frac{(x^2+y^2-1)+2iy}{(x+1)^2+y^2}.$$

$\frac{z-1}{z+1}$ is purely imaginary $\Leftrightarrow x^2 + y^2 - 1 = 0 \Leftrightarrow x^2 + y^2 = 1 \Leftrightarrow |z| = 1.$

9. Let $z_1 = x_1 + iy_1$. Then, $|z_1| = 1 \Rightarrow |z_1|^2 = 1 \Rightarrow x_1^2 + y_1^2 = 1.$

$$z_2 = \frac{z_1 - 1}{z_1 + 1} = \frac{(x_1 + iy_1) - 1}{(x_1 + iy_1) + 1} = \frac{(x_1 - 1) + iy_1}{(x_1 + 1) + iy_1} \times \frac{(x_1 + 1) - iy_1}{(x_1 + 1) - iy_1}$$

$= \frac{(x_1^2 + y_1^2 - 1) + 2iy_1}{(x_1 + 1)^2 + y_1^2} = \frac{2iy_1}{(x_1 + 1)^2 + y_1^2}$, which is purely imaginary. $[\because |z_1|^2 = 1]$

12. We have

$\frac{1 - ix}{1 + ix} = \frac{a - ib}{1} \Rightarrow \frac{(1 - ix) + (1 + ix)}{(1 - ix) - (1 + ix)} = \frac{(a - ib) + 1}{(a - ib) - 1}$ [by componendo and dividendo]

$\Rightarrow \frac{2}{-2ix} = \frac{(a+1) - ib}{(a-1) - ib} \Rightarrow -ix = \frac{(a-1) - ib}{(a+1) - ib} \times \frac{(a+1) + ib}{(a+1) + ib}$

$\Rightarrow -ix = \frac{(a^2 - 1 + b^2) + \{(a-1)b - (a+1)b\}i}{(a+1)^2 + b^2}$

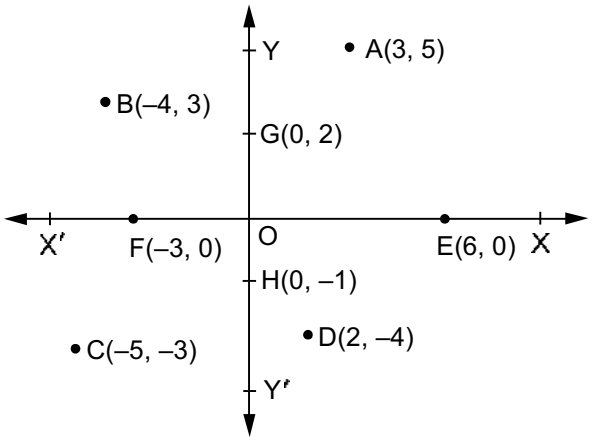
$= \frac{-2bi}{(a+1)^2 + b^2}$ $[\because a^2 + b^2 = 1]$

$\Rightarrow x = \frac{2b}{(a+1)^2 + b^2}$, which is purely real.

POLAR REPRESENTATION OF COMPLEX NUMBERS

COMPLEX PLANE OR ARGAND PLANE

Let $X'OX$ and YOY' be the mutually perpendicular lines, known as the *x*-axis and the *y*-axis respectively. The complex number $(x+iy)$ corresponds to the ordered pair (x, y) and it can be represented by the point $P(x, y)$ in the *x*-*y* plane. The *x*-*y* plane is known as the *complex plane*, or the *Argand plane*. *x*-axis is called the *real axis* and *y*-axis is called the *imaginary axis*.



Note that every number on the *x*-axis is a *real number*, while each on the *y*-axis is an *imaginary number*.

The complex numbers represented geometrically in the above diagram are

$$(3+5i), (-4+3i), (-5-3i), (2-4i),$$

$$(6+0i), (-3+0i), (0+2i) \text{ and } (0-i),$$

represented by the points

$$A(3, 5), B(-4, 3), C(-5, -3), D(2, -4), E(6, 0),$$

$$F(-3, 0), G(0, 2) \text{ and } H(0, -1) \text{ respectively.}$$

POLAR FORM OF A COMPLEX NUMBER

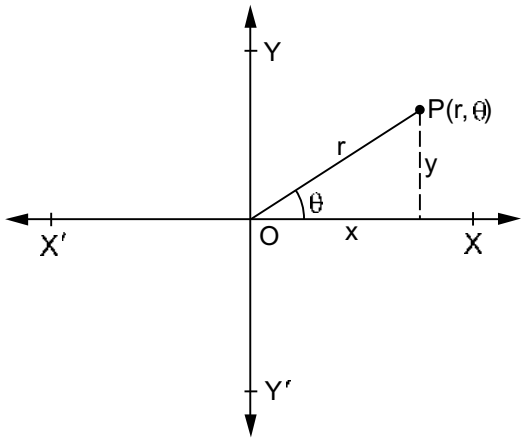
Let the complex number $z = x + iy$ be represented by the point $P(x, y)$ in the complex plane.

Let $\angle XOP = \theta$ and $|OP| = r > 0$. Then, $P(r, \theta)$ are called the *polar coordinates* of *P*.

We call the origin *O* as *pole*.

Clearly, $x = r\cos\theta$ and $y = r\sin\theta$

$\therefore \quad z = r(\cos\theta + i\sin\theta)$.



This is called the *polar form,* or *trigonometric form,* or *modulus-amplitude form,* of z.

Here, $r = \sqrt{x^2 + y^2} = |z|$ is called the *modulus of z*.

And θ is called the *argument,* or *amplitude of z*, written as arg(z), or amp(z).

The value of θ such that $-\pi < \theta \leq \pi$ is called the *principal argument of z*.

METHOD FOR FINDING THE PRINCIPAL ARGUMENT OF A COMPLEX NUMBER

Case I *When $z = (x + iy)$ lies on one of the axes:*

I. *When z is purely real.*

In this case, z lies on the x-axis.

(i) If z lies on positive side of the x-axis then $\theta = 0$.

(ii) If z lies on negative side of the x-axis then $\theta = \pi$.

II. *When z is purely imaginary.*

In this case, z lies on the y-axis.

(i) If z lies on the y-axis and above the x-axis then $\theta = \frac{\pi}{2}$.

(ii) If z lies on the y-axis and below the x-axis then $\theta = \frac{-\pi}{2}$.

Case II *When $z = (x + iy)$ does not lie on any axes:*

Step 1. Find the acute angle α given by $\tan \alpha = \left| \frac{\text{Im}(z)}{\text{Re}(z)} \right|$.

Step 2. Find the quadrant in which $P(x, y)$ lies.

Then, $\theta = \arg(z)$ or $\text{amp}(z)$ may be obtained as under.

(i) *When z lies in quad. I;*

then, $\theta = \alpha \Rightarrow \arg(z) = \alpha$.

(ii) *When z lies in quad. II;*

then, $\theta = (\pi - \alpha) \Rightarrow \arg(z) = (\pi - \alpha)$.

(iii) *When z lies in quad. III;*

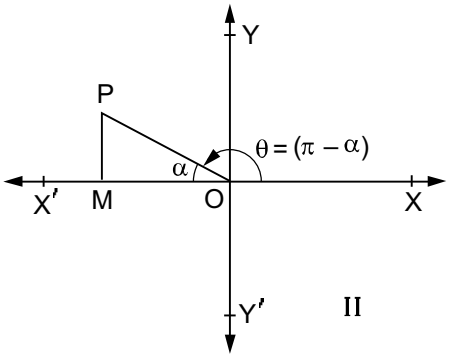
then, $\theta = (\alpha - \pi)$ or $(\pi + \alpha)$

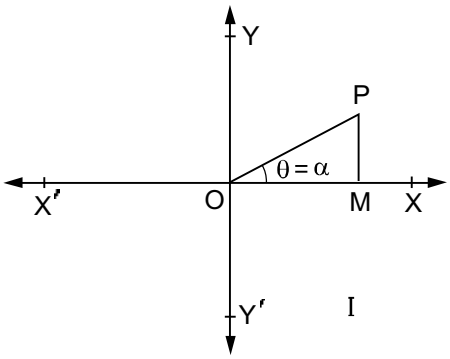
$\Rightarrow \arg(z) = (\alpha - \pi)$ or $(\pi + \alpha)$.

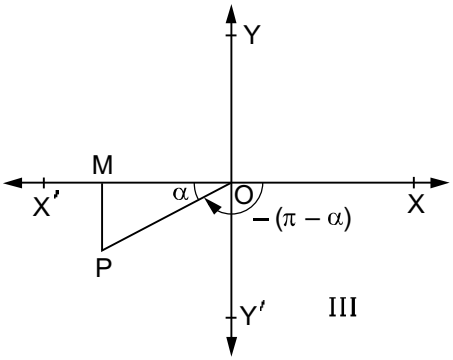
(iv) *When z lies in quad. IV;*

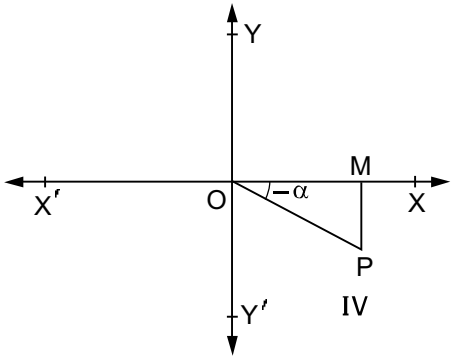
then, $\theta = -\alpha$ or $\theta = (2\pi - \alpha)$

$\Rightarrow \arg(z) = -\alpha$ or $(2\pi - \alpha)$.









SUMMARY

Polar form of $z = x + iy$ is $r(\cos\theta + i\sin\theta)$.

(i) $r = |z| = \sqrt{x^2 + y^2}$.

(ii) $\tan\alpha = \left|\frac{y}{x}\right| = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$.

(iii) When $-\pi < \theta \leq \pi$, then θ is the principal argument of z.

(iv)

Quad. in which z lies	$\arg(z)$
I	$\theta = \alpha$
II	$\theta = \pi - \alpha$
III	$\theta = -(\pi - \alpha)$
IV	$\theta = -\alpha$ or $(2\pi - \alpha)$

(v) *When z is purely real;*

then, z lies on the x-axis.

$(x > 0 \Rightarrow \theta = 0)$ and $(x < 0 \Rightarrow \theta = \pi)$.

(vi) *When z is purely imaginary;*

then, z lies on the y-axis.

$\left(y > 0 \Rightarrow \theta = \frac{\pi}{2}\right)$ and $\left(y < 0 \Rightarrow \theta = \frac{-\pi}{2}\right)$.

SOLVED EXAMPLES

EXAMPLE 1 *Convert each of the following complex numbers into polar form:*

(i) 3 *(ii)* -5 *(iii)* i *(iv)* $-2i$

SOLUTION (i) The given complex number is $z = 3 + 0i$.

Let its polar form be $z = r(\cos\theta + i\sin\theta)$.

Now, $r = |z| = \sqrt{3^2 + 0^2} = \sqrt{9} = 3$.

Clearly, $z = 3 + 0i$ is represented by the point $P(3, 0)$, which lies on the positive side of the x-axis.

$\therefore \quad \arg(z) = \theta = 0$.

Thus, $r = 3$ and $\theta = 0$.

Hence, the required polar form of $z = 3 + 0i$ is $3(\cos 0 + i\sin 0)$.

(ii) The given complex number is $z = -5 + 0i$.

Let its polar form be $z = r(\cos\theta + i\sin\theta)$.

Now, $r = |z| = \sqrt{(-5)^2 + 0^2} = \sqrt{25} = 5$.

Clearly, $z = -5 + 0i$ is represented by the point $P(-5, 0)$, which lies on the negative side of the x-axis.

$\therefore \quad \arg(z) = \pi \Rightarrow \theta = \pi$.

Thus, $r = 5$ and $\theta = \pi$.

Hence, the required polar form of $z = -5 + 0i$ is $5(\cos\pi + i\sin\pi)$.

(iii) The given complex number is $z = 0 + i$.

Let its polar form be $z = r(\cos\theta + i\sin\theta)$.

Now, $r = |z| = \sqrt{0^2 + 1^2} = 1$.

Clearly, $z = 0 + i$ is represented by the point $P(0, 1)$, which lies on the y-axis and above the x-axis.

$\therefore \quad \arg(z) = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{2}$.

Thus, $r = 1$ and $\theta = \frac{\pi}{2}$.

Hence, the required polar form of $z = 0 + i$ is

$$1\cdot\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right), \text{ i.e., } \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right).$$

(iv) The given complex number is $z = 0 - 2i$.

Let its polar form be $z = r(\cos\theta + i\sin\theta)$.

Now, $r = |z| = \sqrt{0^2 + (-2)^2} = \sqrt{4} = 2$.

Clearly, $z = (0 - 2i)$ is represented by the point $P(0, -2)$, which lies on the y-axis and below the x-axis.

$\therefore \quad \arg(z) = \frac{-\pi}{2} \Rightarrow \theta = \frac{-\pi}{2}$.

Thus, $r = 2$ and $\theta = \frac{-\pi}{2}$.

Hence, the required polar form of $z = 0 - 2i$ is

$$z = 2\left\{\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right\}, \text{ i.e., } 2\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right).$$

EXAMPLE 2 *Find the modulus and argument of each of the complex numbers given below:*
(i) $1 + i$ *(ii)* $-\sqrt{3} + i$ *(iii)* $-1 - i\sqrt{3}$

SOLUTION (i) Let $z = 1 + i$. Then, $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$.

Let α be the acute angle given by

$$\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right| = \left|\frac{1}{1}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}.$$

Clearly, the point representing $z = 1 + i$ is $P(1, 1)$, which lies in the first quadrant.

$$\therefore \quad \arg(z) = \theta = \alpha = \frac{\pi}{4}.$$

Hence, $|z| = \sqrt{2}$ and $\arg(z) = \frac{\pi}{4}$.

(ii) Let $z = -\sqrt{3} + i$. Then, $|z| = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$.

Let α be the acute angle given by

$$\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right| = \left|\frac{1}{-\sqrt{3}}\right| = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}.$$

Now, $z = (-\sqrt{3} + i)$ is represented by the point $P(-\sqrt{3}, 1)$, which lies in the second quadrant.

$$\therefore \quad \arg(z) = \theta = (\pi - \alpha) = \left(\pi - \frac{\pi}{6}\right) = \frac{5\pi}{6}.$$

Hence, $|z| = 2$ and $\arg(z) = \frac{5\pi}{6}$.

(iii) Let $z = -1 - i\sqrt{3}$. Then, $|z| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = \sqrt{4} = 2$.

Let α be the acute angle given by

$$\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right| = \left|\frac{-\sqrt{3}}{-1}\right| = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}.$$

Now, $z = (-1 - i\sqrt{3})$ is represented by the point $P(-1, -\sqrt{3})$, which lies in the third quadrant.

$$\therefore \quad \arg(z) = \theta = (\alpha - \pi) = \left(\frac{\pi}{3} - \pi\right) = \frac{-2\pi}{3}.$$

Hence, $|z| = 2$ and $\arg(z) = \frac{-2\pi}{3}$.

EXAMPLE 3 *Convert the complex number* $(1 + i\sqrt{3})$ *into polar form.*

SOLUTION The given complex number is $z = (1 + i\sqrt{3})$.

Let its polar form be $z = r(\cos\theta + i\sin\theta)$.

Now, $r = |z| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$.

Let α be the acute angle, given by

$$\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right| = \left|\frac{\sqrt{3}}{1}\right| = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}.$$

Clearly, the point representing $z = (1 + i\sqrt{3})$ is $P(1, \sqrt{3})$, which lies in the first quadrant.

$$\therefore \quad \arg(z) = \theta = \alpha = \frac{\pi}{3}.$$

Thus, $r = |z| = 2$ and $\theta = \arg(z) = \frac{\pi}{3}$.

Hence, the required polar form of $z = (1 + i\sqrt{3})$ is $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$.

EXAMPLE 4 *Convert the complex number* $(2\sqrt{3} - 2i)$ *into polar form.*

SOLUTION The given complex number is $z = (2\sqrt{3} - 2i)$.

Let its polar form be $z = r(\cos\theta + i\sin\theta)$.

Now, $r = |z| = \sqrt{(2\sqrt{3})^2 + (-2)^2} = \sqrt{12 + 4} = \sqrt{16} = 4$.

Let α be the acute angle, given by

$$\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right| = \left|\frac{-2}{2\sqrt{3}}\right| = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}.$$

Clearly, the point representing $z = (2\sqrt{3} - 2i)$ is $P(2\sqrt{3}, -2)$, which lies in the fourth quadrant.

$$\therefore \quad \arg(z) = \theta = -\alpha = \frac{-\pi}{6}.$$

Thus, $r = |z| = 4$ and $\theta = \arg(z) = \frac{-\pi}{6}$.

Hence, the required polar form of $z = (2\sqrt{3} - 2i)$ is

$$4\left[\cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)\right] = 4\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right).$$

EXAMPLE 5 *Convert the complex number* $(-2 + 2i\sqrt{3})$ *into polar form.*

SOLUTION The given complex number is $z = (-2 + 2i\sqrt{3})$.

Let its polar form be $z = r(\cos\theta + i\sin\theta)$.

Now, $r = |z| = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{4 + 12} = \sqrt{16} = 4$.

Let α be the acute angle, given by

$$\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right| = \left|\frac{2\sqrt{3}}{-2}\right| = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}.$$

Clearly, the point representing $z = (-2 + 2i\sqrt{3})$ is $P(-2, 2\sqrt{3})$, which lies in the second quadrant.

$$\therefore \quad \arg(z) = \theta = (\pi - \alpha) = \left(\pi - \frac{\pi}{3}\right) = \frac{2\pi}{3}.$$

Thus, $r = |z| = 4$ and $\theta = \frac{2\pi}{3}$.

Hence, the required polar form is $z = 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$.

EXAMPLE 6 *Convert* $\dfrac{(1+7i)}{(2-i)^2}$ *into polar form.*

SOLUTION Let $z = \dfrac{(1+7i)}{(2-i)^2} = \dfrac{(1+7i)}{(4+i^2-4i)} = \dfrac{(1+7i)}{(3-4i)} \times \dfrac{(3+4i)}{(3+4i)}$

$$\Rightarrow \quad z = \frac{(1+7i)(3+4i)}{(9+16)} = \frac{-25+25i}{25} = (-1+i).$$

Let its polar form be $z = r(\cos\theta + i\sin\theta)$.

Now, $r = |z| = \sqrt{(-1)^2+1^2} = \sqrt{2}$.

Let α be the acute angle, given by

$$\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right| = \left|\frac{1}{-1}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}.$$

Clearly, the point representing $z = (-1+i)$ is $P(-1, 1)$, which lies in the second quadrant.

$$\therefore \quad \arg(z) = \theta = (\pi - \alpha) = \left(\pi - \frac{\pi}{4}\right) = \frac{3\pi}{4}.$$

Thus, $r = |z| = \sqrt{2}$ and $\theta = \dfrac{3\pi}{4}$.

Hence, the required polar form is $z = \sqrt{2}\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)$.

EXAMPLE 7 *Convert the complex number* $\dfrac{1}{(1+i)}$ *into polar form.*

SOLUTION Let $z = \dfrac{1}{(1+i)} \times \dfrac{(1-i)}{(1-i)} = \dfrac{(1-i)}{(1-i^2)} = \dfrac{(1-i)}{2} = \left(\dfrac{1}{2} - \dfrac{1}{2}i\right)$.

Let its polar form be $z = r(\cos\theta + i\sin\theta)$.

Now, $r = |z| = \sqrt{\left(\dfrac{1}{2}\right)^2 + \left(\dfrac{-1}{2}\right)^2} = \sqrt{\dfrac{1}{4}+\dfrac{1}{4}} = \sqrt{\dfrac{1}{2}} = \dfrac{1}{\sqrt{2}}$.

Let α be the acute angle, given by

$$\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right| = \left|\frac{(-1/2)}{(1/2)}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}.$$

Clearly, the point representing $z = \left(\dfrac{1}{2} - \dfrac{1}{2}i\right)$ is $P\left(\dfrac{1}{2}, \dfrac{-1}{2}\right)$, which lies in the fourth quadrant.

$$\therefore \quad \arg(z) = \theta = -\alpha = -\frac{\pi}{4}.$$

Thus, $r = |z| = \dfrac{1}{\sqrt{2}}$ and $\theta = \dfrac{-\pi}{4}$.

Hence, the required polar form is

$$z = \frac{1}{\sqrt{2}}\left\{\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right\}, \text{ i.e., } \frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right).$$

EXAMPLE 8 *Convert the complex number* $\frac{(i-1)}{\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}$ *into polar form.*

SOLUTION Let $z=\frac{(i-1)}{\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}=\frac{(i-1)}{\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)}=\frac{2(i-1)}{(1+i\sqrt{3})}$

$$\Rightarrow \quad z=\frac{2(i-1)}{(1+i\sqrt{3})}\times\frac{(1-i\sqrt{3})}{(1-i\sqrt{3})}=\frac{(i-1)(1-i\sqrt{3})}{2}$$

$$=\left\{\frac{(\sqrt{3}-1)}{2}+\frac{(\sqrt{3}+1)}{2}i\right\}.$$

Let its polar form be $z=r(\cos\theta+i\sin\theta)$.

Now, $r^2=|z|^2=\left\{\frac{(\sqrt{3}-1)^2}{4}+\frac{(\sqrt{3}+1)^2}{4}\right\}=\frac{8}{4}=2 \Rightarrow r=\sqrt{2}$.

Let α be the acute angle, given by

$$\tan\alpha=\left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|=\left|\frac{(\sqrt{3}+1)}{2}\times\frac{2}{(\sqrt{3}-1)}\right|=\frac{(\sqrt{3}+1)}{(\sqrt{3}-1)}=\frac{\left(1+\frac{1}{\sqrt{3}}\right)}{\left(1-\frac{1}{\sqrt{3}}\right)}$$

$$\Rightarrow \quad \tan\alpha=\frac{\left(\tan\frac{\pi}{4}+\tan\frac{\pi}{6}\right)}{\left(1-\tan\frac{\pi}{4}\cdot\tan\frac{\pi}{6}\right)}=\tan\left(\frac{\pi}{4}+\frac{\pi}{6}\right)=\tan\frac{5\pi}{12}$$

$$\Rightarrow \quad \alpha=\frac{5\pi}{12}.$$

Clearly, the point representing the given complex number is $P\left(\frac{\sqrt{3}-1}{2},\frac{\sqrt{3}+1}{2}\right)$, which lies in the first quadrant.

$$\therefore \quad \arg(z)=\theta=\alpha=\frac{5\pi}{12} \Rightarrow \theta=\frac{5\pi}{12}.$$

Thus, $r=|z|=\sqrt{2}$ and $\theta=\frac{5\pi}{12}$.

Hence, the required polar form is $z=\sqrt{2}\left(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}\right)$.

EXAMPLE 9 *Express the complex number* $(-1-i)$ *in polar form.*

SOLUTION The given complex number is $z=-1-i$.

Let its polar form be $z=r(\cos\theta+i\sin\theta)$.

Now, $r=|z|=\sqrt{(-1)^2+(-1)^2}=\sqrt{2}$.

Let α be the acute angle, given by

$$\tan\alpha=\left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|=\left|\frac{-1}{-1}\right|=1 \Rightarrow \alpha=\frac{\pi}{4}.$$

Clearly, the point representing the complex number $z = -1 - i$ is $P(-1, -1)$, which lies in the third quadrant.

$$\therefore \quad \arg(z) = \theta = -(\pi - \alpha) = -\left(\pi - \frac{\pi}{4}\right) = \frac{-3\pi}{4}.$$

Thus, $r = |z| = \sqrt{2}$ and $\theta = \frac{-3\pi}{4}$.

Hence, the required polar form of $z = (-1 - i)$ is given by

$$z = \sqrt{2}\left\{\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right\}, \text{ i.e., } \sqrt{2}\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right).$$

EXAMPLE 10 *Express the complex number* $(-\sqrt{3} - i)$ *in polar form.*

SOLUTION The given complex number is $z = (-\sqrt{3} - i)$.

Let its polar form be $z = r(\cos\theta + i\sin\theta)$.

Now, $r = |z| = \sqrt{(-\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$.

Let α be the acute angle, given by

$$\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right| = \left|\frac{-1}{-\sqrt{3}}\right| = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}.$$

Clearly, the point representing the complex number $z = (-\sqrt{3} - i)$ is $P(-\sqrt{3}, -1)$, which lies in the third quadrant.

$$\therefore \quad \arg(z) = \theta = -(\pi - \alpha) = -\left(\pi - \frac{\pi}{6}\right) = \frac{-5\pi}{6}.$$

Thus, $r = |z| = 2$ and $\theta = \frac{-5\pi}{6}$.

Hence, the polar form of $z = (-\sqrt{3} - i)$ is given by

$$z = 2\left\{\cos\left(\frac{-5\pi}{6}\right) + i\sin\left(\frac{-5\pi}{6}\right)\right\}, \text{ i.e., } 2\left(\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}\right).$$

EXAMPLE 11 *Convert the complex number* $\frac{1+2i}{1-3i}$ *into polar form.*

SOLUTION Let $z = \frac{(1+2i)}{(1-3i)} \times \frac{(1+3i)}{(1+3i)} = \frac{(1+2i)(1+3i)}{(1+9)} = \frac{-5+5i}{10} = \left(\frac{-1}{2} + \frac{1}{2}i\right)$.

Let its polar form be $z = r(\cos\theta + i\sin\theta)$.

Now, $r = |z| = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$.

Let α be the acute angle, given by

$$\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right| = \left|\frac{(1/2)}{-(1/2)}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}.$$

Clearly, the point representing $z = \left(\frac{-1}{2} + \frac{1}{2}i\right)$ is $P\left(\frac{-1}{2}, \frac{1}{2}\right)$, which lies in the second quadrant.

$\therefore \quad \arg(z) = \theta = (\pi - \alpha) = \left(\pi - \frac{\pi}{4}\right) = \frac{3\pi}{4}.$

Thus, $r = |z| = \frac{1}{\sqrt{2}}$ and $\theta = \frac{3\pi}{4}.$

Hence, the required polar form is $z = \frac{1}{\sqrt{2}}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right).$

EXAMPLE 12 *Express the complex number* $\sin\frac{\pi}{5} + i\left(1 - \cos\frac{\pi}{5}\right)$ *in polar form.*

SOLUTION Let $z = \sin\frac{\pi}{5} + i\left(1 - \cos\frac{\pi}{5}\right).$

Let its polar form be $z = r(\cos\theta + i\sin\theta).$

Now, $r^2 = |z|^2 = \sin^2\frac{\pi}{5} + \left(1 - \cos\frac{\pi}{5}\right)^2 = \left(\sin^2\frac{\pi}{5} + \cos^2\frac{\pi}{5}\right) + 1 - 2\cos\frac{\pi}{5}$

$\Rightarrow \quad r^2 = 2\left(1 - \cos\frac{\pi}{5}\right) = 4\sin^2\frac{\pi}{10} \Rightarrow r = 2\sin\frac{\pi}{10}.$

Let α be the acute angle, given by

$$\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right| = \left|\frac{1 - \cos\frac{\pi}{5}}{\sin\frac{\pi}{5}}\right| = \frac{2\sin^2\frac{\pi}{10}}{2\sin\frac{\pi}{10}\cdot\cos\frac{\pi}{10}} = \tan\frac{\pi}{10} \Rightarrow \alpha = \frac{\pi}{10}.$$

Clearly, the point representing z lies in the first quadrant as $x > 0$ and $y > 0$.

$\therefore \quad \arg(z) = \theta = \alpha = \frac{\pi}{10}.$

Thus, $r = 2\sin\frac{\pi}{10}$ and $\theta = \frac{\pi}{10}.$

Hence, the required polar form is $2\sin\frac{\pi}{10}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right).$

EXAMPLE 13 *Convert* $4(\cos 300^\circ + i\sin 300^\circ)$ *into Cartesian form.*

SOLUTION
$$\begin{aligned} 4(\cos 300^\circ + i\sin 300^\circ) &= 4[\cos(360^\circ - 60^\circ) + i\sin(360^\circ - 60^\circ)] \\ &= 4[\cos(-60^\circ) + i\sin(-60^\circ)] \\ &= 4[\cos 60^\circ - i\sin 60^\circ] = 4\left(\frac{1}{2} - i\cdot\frac{\sqrt{3}}{2}\right) \\ &= (2 - 2i\sqrt{3}). \end{aligned}$$

EXAMPLE 14 *For any complex numbers* z, z_1 *and* z_2 *prove that:*

(i) $\arg(\bar{z}) = -\arg(z)$

(ii) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

(iii) $\arg(z_1\bar{z}_2) = \arg(z_1) - \arg(z_2)$

(iv) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

SOLUTION (i) Let $z = r(\cos\theta + i\sin\theta)$. Then, $|z| = r$ and $\arg(z) = \theta$.

Now, $z = r(\cos\theta + i\sin\theta)$

$\Rightarrow \quad z = r\cos\theta + i(r\sin\theta)$

$\Rightarrow \quad \bar{z} = r\cos\theta - i(r\sin\theta) = r(\cos\theta - i\sin\theta)$

$= r\{\cos(-\theta) + i\sin(-\theta)\}$

$\Rightarrow \quad |\bar{z}| = r$ and $\arg(\bar{z}) = -\theta = -\arg(z)$.

Hence, $\arg(\bar{z}) = -\arg(z)$.

(ii) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then,

$|z_1| = r_1$, $\arg(z_1) = \theta$ and $|z_2| = r_2$, $\arg(z_2) = \theta_2$.

$\therefore \quad z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)$

$+ i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\}$

$= r_1 r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$

$\Rightarrow \quad \arg(z_1 z_2) = (\theta_1 + \theta_2) = \arg(z_1) + \arg(z_2)$.

REMARKS (I) Note here that $|z_1 z_2| = r_1 r_2 = |z_1|\ |z_2|$.

(II) In general, we have

$|z_1 z_2 \ldots z_n| = |z_1|\cdot|z_2|\ldots|z_n|$

and $\arg(z_1 z_2 \ldots z_n) = \arg(z_1) + \arg(z_2) + \ldots + \arg(z_n)$.

(iii) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then,

$\bar{z}_2 = \overline{r_2\cos\theta_2 + i(r_2\sin\theta_2)} = r_2\cos\theta_2 - i(r_2\sin\theta_2)$

$\Rightarrow \quad \bar{z}_2 = r_2\{\cos(-\theta_2) + i\sin(-\theta_2)\}$.

$\therefore \quad z_1\bar{z}_2 = r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2\{\cos(-\theta_2) + i\sin(-\theta_2)\}$

$= r_1 r_2(\cos\theta_1 + i\sin\theta_1)\{\cos(-\theta_2) + i\sin(-\theta_2)\}$

$= r_1 r_2[\cos\{\theta_1 + (-\theta_2)\} + i\sin\{\theta_1 + (-\theta_2)\}]$

$= r_1 r_2\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}$.

Hence, $\arg(z_1\bar{z}_2) = (\theta_1 - \theta_2) = \arg(z_1) - \arg(z_2)$.

(iv) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then,

$|z_1| = r_1$, $|z_2| = r_2$, $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$.

$\therefore \quad \dfrac{z_1}{z_2} = \dfrac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \dfrac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$

$= \dfrac{r_1}{r_2}\cdot\left\{\dfrac{(\cos\theta_1\cdot\cos\theta_2 - \sin\theta_1\cdot\sin\theta_2) + i(\sin\theta_1\cdot\cos\theta_2 - \cos\theta_1\cdot\sin\theta_2)}{(\cos^2\theta_2 + \sin^2\theta_2)}\right\}$

$= \dfrac{r_1}{r_2}\cdot\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}$

$\Rightarrow \quad \arg\left(\dfrac{z_1}{z_2}\right) = (\theta_1 - \theta_2) = \arg(z_1) - \arg(z_2)$.

Hence, $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$.

EXERCISE 5D

Find the modulus and argument of each of the following complex numbers and hence express each of them in polar form:

1. 4 **2.** -2 **3.** $-i$ **4.** $2i$

5. $1-i$ **6.** $-1+i$ **7.** $\sqrt{3}+i$ **8.** $-1+\sqrt{3}\,i$

9. $1-\sqrt{3}\,i$ **10.** $2-2i$ **11.** $-4+4\sqrt{3}\,i$ **12.** $-3\sqrt{2}+3\sqrt{2}\,i$

13. $\dfrac{1+i}{1-i}$ **14.** $\dfrac{1-i}{1+i}$ **15.** $\dfrac{1+3i}{1-2i}$ **16.** $\dfrac{1-3i}{1+2i}$

17. $\dfrac{5-i}{2-3i}$ **18.** $\dfrac{-16}{1+\sqrt{3}\,i}$ **19.** $\dfrac{2+6\sqrt{3}\,i}{5+\sqrt{3}\,i}$ **20.** $\sqrt{\dfrac{1+i}{1-i}}$

21. $-\sqrt{3}-i$ **22.** $(i^{25})^3$ **23.** $\dfrac{(1-i)}{\left(\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}\right)}$

24. $(\sin 120^\circ - i\cos 120^\circ)$

ANSWERS (EXERCISE 5D)

1. $4, 0, 4(\cos 0 + i\sin 0)$

2. $2, \pi, 2(\cos\pi + i\sin\pi)$

3. $1, \dfrac{-\pi}{2}, \left\{\cos\left(\dfrac{-\pi}{2}\right)+i\sin\left(\dfrac{-\pi}{2}\right)\right\}$

4. $2, \dfrac{\pi}{2}, 2\left(\cos\dfrac{\pi}{2}+i\sin\dfrac{\pi}{2}\right)$

5. $\sqrt{2}, \dfrac{-\pi}{4}, \sqrt{2}\left\{\cos\left(\dfrac{-\pi}{4}\right)+i\sin\left(\dfrac{-\pi}{4}\right)\right\}$

6. $\sqrt{2}, \dfrac{3\pi}{4}, \sqrt{2}\left(\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right)$

7. $2, \dfrac{\pi}{6}, 2\left(\cos\dfrac{\pi}{6}+i\sin\dfrac{\pi}{6}\right)$

8. $2, \dfrac{2\pi}{3}, 2\left(\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}\right)$

9. $2, \dfrac{-\pi}{3}, 2\left\{\cos\left(\dfrac{-\pi}{3}\right)+i\sin\left(\dfrac{-\pi}{3}\right)\right\}$

10. $2\sqrt{2}, \dfrac{-\pi}{4}, 2\sqrt{2}\left\{\cos\left(\dfrac{-\pi}{4}\right)+i\sin\left(\dfrac{-\pi}{4}\right)\right\}$

11. $8, \dfrac{2\pi}{3}, 8\left\{\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}\right\}$

12. $6, \dfrac{3\pi}{4}, 6\left(\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right)$

13. $1, \dfrac{\pi}{2}, \left(\cos\dfrac{\pi}{2}+i\sin\dfrac{\pi}{2}\right)$

14. $1, \dfrac{-\pi}{2}, \left\{\cos\left(\dfrac{-\pi}{2}\right)+i\sin\left(\dfrac{-\pi}{2}\right)\right\}$

15. $\sqrt{2}, \dfrac{3\pi}{4}, \sqrt{2}\left(\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right)$

16. $\sqrt{2}, \dfrac{-3\pi}{4}, \left\{\cos\left(\dfrac{-3\pi}{4}\right)+i\sin\left(\dfrac{-3\pi}{4}\right)\right\}$

17. $\sqrt{2}, \dfrac{\pi}{4}, \sqrt{2}\left(\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4}\right)$

18. $8, \dfrac{2\pi}{3}, 8\left(\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}\right)$

19. $2, \dfrac{\pi}{3}, 2\left(\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}\right)$

20. $1, \dfrac{\pi}{4}, \left(\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4}\right)$

21. $2, \dfrac{-5\pi}{6}, 2\left\{\cos\left(\dfrac{-5\pi}{6}\right)+i\sin\left(\dfrac{-5\pi}{6}\right)\right\}$

22. $1, \dfrac{-\pi}{2}, \left\{\cos\left(\dfrac{-\pi}{2}\right)+i\sin\left(\dfrac{-\pi}{2}\right)\right\}$

23. $\sqrt{2}, \dfrac{7\pi}{12}, \sqrt{2}\left(\cos\dfrac{7\pi}{12}-i\sin\dfrac{7\pi}{12}\right)$

24. $(\cos 30^\circ + i\sin 30^\circ)$

HINTS TO SOME SELECTED QUESTIONS

21. Let $z = -\sqrt{3} - i \Rightarrow r^2 = |z|^2 = (-\sqrt{3})^2 + (-1)^2 = 4 \Rightarrow r = |z| = 2.$

$$\tan\alpha = \left|\frac{-1}{-\sqrt{3}}\right| = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}.$$

The given number represents the point $P(-\sqrt{3}, -1)$ which lies in the third quadrant.

$\therefore \quad \arg(z) = \theta = -(\pi - \alpha) = -\left(\pi - \frac{\pi}{6}\right) = \frac{-5\pi}{6}.$

$\therefore \quad z = 2\left[\cos\left(\frac{-5\pi}{6}\right) + i\sin\left(\frac{-5\pi}{6}\right)\right].$

23. Given number $= \frac{2(1-i)}{(1+\sqrt{3}\,i)} \times \frac{(1-\sqrt{3}\,i)}{(1-\sqrt{3}\,i)} = \left\{\frac{(1-\sqrt{3})}{2} - \frac{(1+\sqrt{3})}{2}i\right\}$

$\therefore \quad |z|^2 = \frac{(1-\sqrt{3})^2}{4} + \frac{(1+\sqrt{3})^2}{4} = \frac{2(1+3)}{4} = 2 \Rightarrow |z| = \sqrt{2}.$

$$\tan\alpha = \left|\frac{\frac{-(1+\sqrt{3})}{2}}{\frac{(1-\sqrt{3})}{2}}\right| = \frac{(\sqrt{3}+1)}{(\sqrt{3}-1)} = \frac{\left(1+\frac{1}{\sqrt{3}}\right)}{\left(1-\frac{1}{\sqrt{3}}\right)} = \frac{\tan\frac{\pi}{4} + \tan\frac{\pi}{6}}{1 - \tan\frac{\pi}{4}\cdot\tan\frac{\pi}{6}}$$

$$= \tan\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \tan\frac{5\pi}{12}.$$

The given number z is represented by the point $P\left(\frac{-(\sqrt{3}-1)}{2}, \frac{-(\sqrt{3}+1)}{2}\right)$.

So, it lies in the third quadrant.

$\therefore \quad \arg(z) = \theta = -(\pi - \alpha) = -\left(\pi - \frac{5\pi}{12}\right) = \frac{-7\pi}{12}.$

Hence, $z = \sqrt{2}\left\{\cos\left(\frac{-7\pi}{12}\right) + i\sin\left(\frac{-7\pi}{12}\right)\right\}.$

QUADRATIC EQUATIONS (With Complex Roots)

FUNDAMENTAL THEOREM OF ALGEBRA

A polynomial equation of degree n has at the most n roots.

SOLVING A QUADRATIC EQUATION

Let the given equation be $ax^2 + bx + c = 0$, where $a, b, c \in R$ and $a \neq 0$.

Then, $ax^2 + bx + c = 0, a \neq 0$

$\Rightarrow \quad ax^2 + bx = -c$

$\Rightarrow \quad x^2 + \frac{b}{a}\cdot x = \frac{-c}{a}$

$\Rightarrow \quad x^2 + \frac{b}{a}\cdot x + \left(\frac{b}{2a}\right)^2 = \frac{-c}{a} + \left(\frac{b}{2a}\right)^2$ [adding $\left(\frac{b}{2a}\right)^2$ on both sides]

$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{(b^2 - 4ac)}{4a^2}$

$\Rightarrow \quad x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$

$$\Rightarrow \quad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Hence, the roots of the equation $ax^2 + bx + c = 0, a \neq 0$ are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The expression $(b^2 - 4ac)$ is called the *discriminant*.

If $(b^2 - 4ac) < 0$, the given quadratic equation will have complex roots.

Here, we shall consider only the quadratic equations having complex roots.

SOLVED EXAMPLES

EXAMPLE 1 *Solve:* $x^2 + 3 = 0.$

SOLUTION We have

$$x^2 + 3 = 0 \Rightarrow x^2 = -3 \Rightarrow x = \pm\sqrt{-3} = \pm i\sqrt{3}.$$

$\therefore$ solution set $= \{i\sqrt{3}, -i\sqrt{3}\}.$

EXAMPLE 2 *Solve:* $x^2 + 3x + 9 = 0.$

SOLUTION The given equation is $x^2 + 3x + 9 = 0.$

This is of the form $ax^2 + bx + c = 0$, where $a = 1, b = 3$ and $c = 9.$

$\therefore \quad (b^2 - 4ac) = (3^2 - 4 \times 1 \times 9) = (9 - 36) = -27 < 0.$

So, the given equation has complex roots.

These roots are given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{-27}}{2 \times 1} \qquad [\because \ b^2 - 4ac = -27]$$

$$= \frac{-3 \pm i\sqrt{27}}{2} = \frac{-3 \pm i3\sqrt{3}}{2}.$$

$\therefore$ solution set $= \left\{\frac{-3 + i3\sqrt{3}}{2}, \frac{-3 - i3\sqrt{3}}{2}\right\}$

$$= \left\{\frac{-3}{2} + \frac{3\sqrt{3}}{2}i, \frac{-3}{2} - \frac{3\sqrt{3}}{2}i\right\}.$$

EXAMPLE 3 *Solve:* $9x^2 + 10x + 3 = 0.$

SOLUTION The given equation is $9x^2 + 10x + 3 = 0.$

This is of the form $ax^2 + bx + c = 0$, where $a = 9, b = 10$ and $c = 3.$

$\therefore \quad (b^2 - 4ac) = \{(10)^2 - 4 \times 9 \times 3\} = (100 - 108) = -8 < 0.$

So, the given equation has complex roots.

These roots are given by

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-10 \pm \sqrt{-8}}{2\times 9} \qquad [\because\ (b^2-4ac) = -8]$$

$$= \frac{-10 \pm i2\sqrt{2}}{18} = \frac{-5 \pm i\sqrt{2}}{9}.$$

$$\therefore \quad \text{solution set} = \left\{\frac{-5+i\sqrt{2}}{9}, \frac{-5-i\sqrt{2}}{9}\right\} = \left\{\frac{-5}{9}+\frac{\sqrt{2}}{9}i, \frac{-5}{9}-\frac{\sqrt{2}}{9}i\right\}.$$

EXAMPLE 4 *Solve:* $\sqrt{2}x^2 + x + \sqrt{2} = 0.$

SOLUTION The given equation is $\sqrt{2}x^2 + x + \sqrt{2} = 0.$

This is of the form $ax^2 + bx + c = 0$, where $a = \sqrt{2}, b = 1$ and $c = \sqrt{2}$.

$$\therefore \quad (b^2-4ac) = (1^2 - 4\times\sqrt{2}\times\sqrt{2}) = (1-8) = -7 < 0.$$

So, the given equation has complex roots.

These roots are given by

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-1 \pm \sqrt{-7}}{2\sqrt{2}} \qquad [\because\ b^2-4ac = -7]$$

$$= \frac{-1 \pm i\sqrt{7}}{2\sqrt{2}}.$$

$$\therefore \quad \text{solution set} = \left\{\frac{-1+i\sqrt{7}}{2\sqrt{2}}, \frac{-1-i\sqrt{7}}{2\sqrt{2}}\right\}$$

$$= \left\{\frac{-1}{2\sqrt{2}}+\frac{\sqrt{7}}{2\sqrt{2}}i, \frac{-1}{2\sqrt{2}}-\frac{\sqrt{7}}{2\sqrt{2}}i\right\}.$$

EXAMPLE 5 *Solve:* $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0.$

SOLUTION The given equation is

$$\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0.$$

This is of the form $ax^2 + bx + c = 0$, where $a = \sqrt{3}, b = -\sqrt{2}$ and $c = 3\sqrt{3}$.

$$\therefore \quad (b^2-4ac) = \{(-\sqrt{2})^2 - 4\times\sqrt{3}\times 3\sqrt{3}\} = (2-36) = -34 < 0.$$

So, the given equation has complex roots.

These roots are given by

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{\sqrt{2} \pm \sqrt{-34}}{2\times\sqrt{3}} \qquad [\because\ b^2-4ac = -34]$$

$$= \frac{\sqrt{2} \pm i\sqrt{34}}{2\sqrt{3}}.$$

$$\therefore \quad \text{solution set} = \left\{\frac{\sqrt{2}+i\sqrt{34}}{2\sqrt{3}}, \frac{\sqrt{2}-i\sqrt{34}}{2\sqrt{3}}\right\}$$

$$= \left\{\frac{1}{\sqrt{6}}+\frac{\sqrt{34}}{2\sqrt{3}}i, \frac{1}{\sqrt{6}}-\frac{\sqrt{34}}{2\sqrt{3}}i\right\}.$$

EXAMPLE 6 *Solve:* $3x^2 + 8ix + 3 = 0.$

SOLUTION The given equation is

$$3x^2 + 8ix + 3 = 0.$$

This is of the form $ax^2 + bx + c = 0$, where $a = 3$, $b = 8i$ and $c = 3$.

$\therefore \quad (b^2 - 4ac) = \{(8i)^2 - 4 \times 3 \times 3\} = (-64 - 36) = -100 < 0.$

So, the given equation has complex roots.

These roots are given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-8i \pm \sqrt{-100}}{2 \times 3} \qquad [\because\ b^2 - 4ac = -100]$$

$$= \frac{-8i \pm 10i}{6} = \frac{-4i \pm 5i}{3}.$$

Thus, the roots of the given equation are

$$\frac{-4i + 5i}{3} = \frac{i}{3} \text{ and } \frac{-4i - 5i}{3} = \frac{-9i}{3} = -3i.$$

$\therefore \quad$ solution set $= \left\{\frac{i}{3}, -3i\right\}.$

EXERCISE 5E

Solve:

1. $x^2 + 2 = 0$
2. $x^2 + 5 = 0$
3. $2x^2 + 1 = 0$
4. $x^2 + x + 1 = 0$
5. $x^2 - x + 2 = 0$
6. $x^2 + 2x + 2 = 0$
7. $2x^2 - 4x + 3 = 0$
8. $x^2 + 3x + 5 = 0$
9. $\sqrt{5}x^2 + x + \sqrt{5} = 0$
10. $25x^2 - 30x + 11 = 0$
11. $8x^2 + 2x + 1 = 0$
12. $27x^2 + 10x + 1 = 0$
13. $2x^2 - \sqrt{3}x + 1 = 0$
14. $17x^2 - 8x + 1 = 0$
15. $3x^2 + 5 = 7x$
16. $3x^2 - 4x + \frac{20}{3} = 0$
17. $3x^2 + 7ix + 6 = 0$
18. $21x^2 - 28x + 10 = 0$
19. $x^2 + 13 = 4x$
20. $x^2 + 3ix + 10 = 0$
21. $2x^2 + 3ix + 2 = 0$

ANSWERS (EXERCISE 5E)

1. $\{i\sqrt{2}, -i\sqrt{2}\}$
2. $\{i\sqrt{5}, -i\sqrt{5}\}$
3. $\left\{\frac{i}{\sqrt{2}}, \frac{-i}{\sqrt{2}}\right\}$
4. $\left\{\frac{-1}{2} + \frac{\sqrt{3}}{2}i, \frac{-1}{2} - \frac{\sqrt{3}}{2}i\right\}$
5. $\left\{\frac{1}{2} + \frac{\sqrt{7}}{2}i, \frac{1}{2} - \frac{\sqrt{7}}{2}i\right\}$
6. $\{-1 + i, -1 - i\}$
7. $\left\{1 + \frac{1}{\sqrt{2}}i, 1 - \frac{1}{\sqrt{2}}i\right\}$
8. $\left\{\frac{-3}{2} + \frac{\sqrt{11}}{2}i, \frac{-3}{2} - \frac{\sqrt{11}}{2}i\right\}$
9. $\left\{\frac{-1}{2\sqrt{5}} + \frac{\sqrt{19}}{2\sqrt{5}}i, \frac{-1}{2\sqrt{5}} - \frac{\sqrt{19}}{2\sqrt{5}}i\right\}$
10. $\left\{\frac{3}{5} + \frac{\sqrt{2}}{5}i, \frac{3}{5} - \frac{\sqrt{2}}{5}i\right\}$

11. $\left\{\frac{-1}{8}+\frac{\sqrt{7}}{8}i, \frac{-1}{8}-\frac{\sqrt{7}}{8}i\right\}$ **12.** $\left\{\frac{-5}{27}+\frac{\sqrt{2}}{27}i, \frac{-5}{27}-\frac{\sqrt{2}}{27}i\right\}$

13. $\left\{\frac{\sqrt{3}}{4}+\frac{\sqrt{5}}{4}i, \frac{\sqrt{3}}{4}-\frac{\sqrt{5}}{4}i\right\}$ **14.** $\left\{\frac{4}{17}+\frac{1}{17}i, \frac{4}{17}-\frac{1}{17}i\right\}$

15. $\left\{\frac{7}{6}+\frac{\sqrt{11}}{6}i, \frac{7}{6}-\frac{\sqrt{11}}{6}i\right\}$ **16.** $\left\{\frac{2}{3}+\frac{4}{3}i, \frac{2}{3}-\frac{4}{3}i\right\}$

17. $\left\{\frac{2}{3}i, -3i\right\}$ **18.** $\left\{\frac{2}{3}+\frac{\sqrt{14}}{21}i, \frac{2}{3}-\frac{\sqrt{14}}{21}i\right\}$

19. $\{2+3i, 2-3i\}$ **20.** $\{2i, -5i\}$ **21.** $\left\{\frac{1}{2}i, -2i\right\}$

HINTS TO SOME SELECTED QUESTIONS

17. $a = 3, b = 7i$ and $c = 6$.

$\therefore \quad (b^2 - 4ac) = \{(7i)^2 - 4 \times 3 \times 6\} = (-49 - 72) = (-121) < 0.$

So, the given equation has complex roots, given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-7i \pm \sqrt{-121}}{2 \times 3} = \frac{-7i \pm 11i}{6}.$$

Solution set $= \left\{\frac{-7i+11i}{6}, \frac{-7i-11i}{6}\right\} = \left\{\frac{2}{3}i, -3i\right\}.$

SQUARE ROOTS OF A COMPLEX NUMBER

To Evaluate $\sqrt{a+ib}$*:*

SOLUTION Let $\sqrt{a+ib} = (x+iy)$. … (i)

On squaring both sides of (i), we get

$$(a+ib) = (x+iy)^2$$

$$\Rightarrow \quad (a+ib) = (x^2 - y^2) + i(2xy). \quad \text{… (ii)}$$

On comparing real parts and imaginary parts on both sides of (ii), we get

$$(x^2 - y^2) = a \text{ and } 2xy = b$$

$$\Rightarrow \quad (x^2 + y^2) = \sqrt{(x^2 - y^2)^2 + 4x^2y^2} = \sqrt{a^2 + b^2}.$$

$\therefore \quad (x^2 - y^2) = a$ … (iii) and $(x^2 + y^2) = \sqrt{a^2 + b^2}$ … (iv)

On solving (iii) and (iv), we get

$$x^2 = \frac{1}{2}\left\{\sqrt{a^2+b^2} + a\right\} \text{ and } y^2 = \frac{1}{2}\left\{\sqrt{a^2+b^2} - a\right\}.$$

Hence, $\sqrt{a+ib} = \pm\left[\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}+a\right\}} + i\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}-a\right\}}\right].$

REMARK Similarly, by assuming that $\sqrt{a-ib} = (x-iy)$, we may find $\sqrt{a-ib}$.

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate* $\sqrt{6+8i}$.

SOLUTION Let $\sqrt{6+8i} = (x+iy)$. ... (i)

On squaring both sides of (i), we get

$$6+8i = (x+iy)^2 \Rightarrow 6+8i = (x^2-y^2)+i(2xy). \quad \text{... (ii)}$$

On comparing real parts and imaginary parts on both sides of (ii), we get

$$x^2-y^2 = 6 \text{ and } 2xy = 8$$

$$\Rightarrow \quad x^2-y^2 = 6 \text{ and } xy = 4$$

$$\Rightarrow \quad (x^2+y^2) = \sqrt{(x^2-y^2)^2+4x^2y^2} = \sqrt{6^2+4\times 16} = \sqrt{100} = 10$$

$$\Rightarrow \quad x^2-y^2 = 6 \text{ and } x^2+y^2 = 10$$

$$\Rightarrow \quad 2x^2 = 16 \text{ and } 2y^2 = 4$$

$$\Rightarrow \quad x^2 = 8 \text{ and } y^2 = 2$$

$$\Rightarrow \quad x = \pm 2\sqrt{2} \text{ and } y = \pm\sqrt{2}.$$

Since $xy > 0$, so x and y are of the same sign.

$$\therefore \quad (x = 2\sqrt{2} \text{ and } y = \sqrt{2}) \text{ or } (x = -2\sqrt{2} \text{ and } y = -\sqrt{2}).$$

Hence, $\sqrt{6+8i} = (2\sqrt{2}+\sqrt{2}\,i)$ or $(-2\sqrt{2}-\sqrt{2}\,i)$.

EXAMPLE 2 *Evaluate* $\sqrt{-5+12i}$.

SOLUTION Let $\sqrt{-5+12i} = (x+iy)$. ... (i)

On squaring both sides of (i), we get

$$-5+12i = (x+iy)^2 \Rightarrow -5+12i = (x^2-y^2)+i(2xy). \quad \text{... (ii)}$$

On comparing real parts and imaginary parts on both sides of (ii), we get

$$x^2-y^2 = -5 \text{ and } 2xy = 12$$

$$\Rightarrow \quad x^2-y^2 = -5 \text{ and } xy = 6$$

$$\Rightarrow \quad (x^2+y^2) = \sqrt{(x^2-y^2)^2+4x^2y^2} = \sqrt{(-5)^2+4\times 36} = \sqrt{169} = 13$$

$$\Rightarrow \quad x^2-y^2 = -5 \text{ and } x^2+y^2 = 13$$

$$\Rightarrow \quad 2x^2 = 8 \text{ and } 2y^2 = 18$$

$$\Rightarrow \quad x^2 = 4 \text{ and } y^2 = 9$$

$$\Rightarrow \quad x = \pm 2 \text{ and } y = \pm 3.$$

Since $xy > 0$, so x and y are of the same sign.

$$\therefore \quad (x = 2 \text{ and } y = 3) \text{ or } (x = -2 \text{ and } y = -3).$$

Hence, $\sqrt{-5+12i} = (2+3i)$ or $(-2-3i)$.

EXAMPLE 3 *Evaluate* $\sqrt{8-15i}$.

SOLUTION Let $\sqrt{8-15i} = (x - iy)$. … (i)

On squaring both sides of (i), we get

$$(8-15i) = (x-iy)^2 \Rightarrow (8-15i) = (x^2 - y^2) - i(2xy). \quad \text{… (ii)}$$

On comparing real parts and imaginary parts on both sides of (ii), we get

$$x^2 - y^2 = 8 \text{ and } 2xy = 15$$

$$\Rightarrow \quad x^2 - y^2 = 8 \text{ and } xy = \frac{15}{2}$$

$$\Rightarrow \quad (x^2 + y^2) = \sqrt{(x^2-y^2)^2 + 4x^2y^2} = \sqrt{64+225} = \sqrt{289} = 17$$

$$\Rightarrow \quad x^2 - y^2 = 8 \text{ and } x^2 + y^2 = 17$$

$$\Rightarrow \quad 2x^2 = 25 \text{ and } 2y^2 = 9$$

$$\Rightarrow \quad x^2 = \frac{25}{2} \text{ and } y^2 = \frac{9}{2}$$

$$\Rightarrow \quad x = \pm\frac{5}{\sqrt{2}} \text{ and } y = \pm\frac{3}{\sqrt{2}}.$$

Since $xy > 0$, so x and y are of the same sign.

Hence, $\sqrt{8-15i} = \left(\frac{5}{\sqrt{2}} - \frac{3}{2}i\right)$ or $\left(\frac{-5}{\sqrt{2}} + \frac{3}{\sqrt{2}}i\right)$.

EXAMPLE 4 *Evaluate* $\sqrt{-24-10i}$.

SOLUTION Let $\sqrt{-24-10i} = (x - iy)$. … (i)

On squaring both sides of (i), we get

$$(-24-10i) = (x^2 - y^2) - i(2xy). \quad \text{… (ii)}$$

On comparing real parts and imaginary parts on both sides of (ii), we get

$$x^2 - y^2 = -24 \text{ and } 2xy = 10$$

$$\Rightarrow \quad x^2 - y^2 = -24 \text{ and } xy = 5$$

$$\Rightarrow \quad (x^2 + y^2) = \sqrt{(x^2-y^2)^2 + 4x^2y^2} = \sqrt{(-24)^2 + 4\times 25} = \sqrt{676} = 26$$

$$\Rightarrow \quad x^2 - y^2 = -24 \text{ and } x^2 + y^2 = 26$$

$$\Rightarrow \quad 2x^2 = 2 \text{ and } 2y^2 = 50$$

$$\Rightarrow \quad x^2 = 1 \text{ and } y^2 = 25 \Rightarrow x = \pm 1 \text{ and } y = \pm 5.$$

Since $xy > 0$, so x and y are of the same sign.

$$\therefore \quad \sqrt{-24-10i} = (1-5i) \text{ or } (-1+5i).$$

EXAMPLE 5 *Evaluate* $\sqrt{-i}$.

SOLUTION Let $\sqrt{-i} = (x - iy)$. … (i)

On squaring both sides of (i), we get

$$(-i) = (x-iy)^2 \Rightarrow -i = (x^2 - y^2) - i(2xy). \quad \text{… (ii)}$$

On comparing real parts and imaginary parts on both sides of (ii), we get

$$(x^2 - y^2) = 0 \text{ and } 2xy = 1.$$

$$\therefore \quad (x^2 + y^2) = \sqrt{(x^2 - y^2)^2 + 4x^2y^2} = \sqrt{0^2 + 1^2} = \sqrt{0 + 1} = \sqrt{1} = 1.$$

Thus, $(x^2 - y^2) = 0$ … (iii) and $(x^2 + y^2) = 1$ … (iv).

On solving (iii) and (iv), we get $x^2 = \frac{1}{2}$ and $y^2 = \frac{1}{2}$.

$$\therefore \quad x = \pm\frac{1}{\sqrt{2}} \text{ and } y = \pm\frac{1}{\sqrt{2}}.$$

Since $xy > 0$, so x and y are of the same sign.

$$\therefore \quad \left(x = \frac{1}{\sqrt{2}}, y = \frac{1}{\sqrt{2}}\right) \text{ or } \left(x = \frac{-1}{\sqrt{2}}, y = \frac{-1}{\sqrt{2}}\right).$$

Hence, $\sqrt{-i} = \left(\frac{1}{\sqrt{2}} - i \cdot \frac{1}{\sqrt{2}}\right)$ or $\left(\frac{-1}{\sqrt{2}} + i \cdot \frac{1}{\sqrt{2}}\right)$.

EXAMPLE 6 *Evaluate* $\sqrt{4 + 3\sqrt{-20}} + \sqrt{4 - 3\sqrt{-20}}$.

SOLUTION $\sqrt{4 + 3\sqrt{-20}} = \sqrt{4 + 6\sqrt{5}\,i}$ and $\sqrt{4 - 3\sqrt{-20}} = \sqrt{4 - 6\sqrt{5}\,i}$.

Let $\sqrt{4 + 6\sqrt{5}\,i} = (x + iy)$. … (i)

On squaring both sides of (i), we get

$$(4 + 6\sqrt{5}\,i) = (x + iy)^2 \Rightarrow (4 + 6\sqrt{5}\,i) = (x^2 - y^2) + i(2xy). \quad \text{… (ii)}$$

On comparing real parts and imaginary parts on both sides of (ii), we get

$$(x^2 - y^2) = 4 \text{ and } 2xy = 6\sqrt{5}.$$

$$\therefore \quad (x^2 + y^2) = \sqrt{(x^2 - y^2)^2 + 4x^2y^2} = \sqrt{(4)^2 + (6\sqrt{5})^2}$$

$$= \sqrt{16 + 180} = \sqrt{196} = 14.$$

Thus, $(x^2 - y^2) = 4$ … (iii) and $(x^2 + y^2) = 14$ … (iv).

On solving (iii) and (iv), we get $x^2 = 9$ and $y^2 = 5$.

$$\therefore \quad x = \pm 3 \text{ and } y = \pm\sqrt{5}.$$

Since $xy > 0$, so x and y are of the same sign.

$$\therefore \quad (x = 3, y = \sqrt{5}) \text{ or } (x = -3, y = -\sqrt{5}).$$

$$\therefore \quad \sqrt{4 + 3\sqrt{-20}} = \sqrt{4 + 6\sqrt{5}\,i} = (3 + \sqrt{5}\,i) \text{ or } (-3 - \sqrt{5}\,i).$$

Similarly, $\sqrt{4 - 3\sqrt{-20}} = \sqrt{4 - 6\sqrt{5}\,i} = (3 - \sqrt{5}\,i)$ or $(-3 + \sqrt{5}\,i)$.

$$\therefore \quad \sqrt{4 + 3\sqrt{-20}} + \sqrt{4 - 3\sqrt{-20}} = \begin{cases} \{(3 + \sqrt{5}\,i) + (3 - \sqrt{5}\,i)\} = 6 \\ \text{or} \\ \{-3 - \sqrt{5}\,i\} + \{-3 + \sqrt{5}\,i\} = -6. \end{cases}$$

Hence, $\sqrt{4 + 3\sqrt{-20}} + \sqrt{4 - 3\sqrt{-20}} = 6$ or -6.

EXERCISE 5F

Evaluate:

1. $\sqrt{5+12i}$ **2.** $\sqrt{-7+24i}$ **3.** $\sqrt{-2+2\sqrt{3}\,i}$

4. $\sqrt{1+4\sqrt{-3}}$ **5.** $\sqrt{i}$ **6.** $\sqrt{4i}$

7. $\sqrt{3+4\sqrt{-7}}$ **8.** $\sqrt{16-30i}$ **9.** $\sqrt{-4-3i}$

10. $\sqrt{-15-8i}$ **11.** $\sqrt{-11-60i}$ **12.** $\sqrt{7-30\sqrt{-2}}$

13. $\sqrt{-8i}$ **14.** $\sqrt{1-i}$

ANSWERS (EXERCISE 5F)

1. $(3+2i)$ or $(-3-2i)$ **2.** $(3+4i)$ or $(-3-4i)$

3. $(1+\sqrt{3}\,i)$ or $(-1-\sqrt{3}\,i)$ **4.** $(2+\sqrt{3}\,i)$ or $(-2-\sqrt{3}\,i)$

5. $\frac{1}{\sqrt{2}}(1+i)$ or $\frac{1}{\sqrt{2}}(-1-i)$ **6.** $\sqrt{2}(1+i)$ or $\sqrt{2}(-1-i)$

7. $(\sqrt{7}+2i)$ or $(-\sqrt{7}-2i)$ **8.** $(5-3i)$ or $(-5+3i)$

9. $\frac{1}{\sqrt{2}}(1-3i)$ or $\frac{1}{\sqrt{2}}(-1+3i)$ **10.** $(1-4i)$ or $(-1+4i)$

11. $(5-6i)$ or $(-5+6i)$ **12.** $(5-3\sqrt{2}\,i)$ or $(-5+3\sqrt{2}\,i)$

13. $(2-2i), (-2+2i)$

14. $\left\{\sqrt{\frac{\sqrt{2}+1}{2}}\right\}-\left\{\sqrt{\frac{\sqrt{2}-1}{2}}\right\}i$ or $-\left\{\sqrt{\frac{\sqrt{2}+1}{2}}\right\}+\left\{\sqrt{\frac{\sqrt{2}-1}{2}}\right\}i$

HINTS TO SOME SELECTED QUESTIONS

14. Let $\sqrt{1-i}=x-iy.$ … (i)

On squaring both sides, we get

$$(1-i)=(x-iy)^2=(x^2-y^2)-i(2xy)$$

$\therefore \quad (x^2-y^2)=1$ and $2xy=1$

$\Rightarrow \quad (x^2+y^2)=\sqrt{(x^2-y^2)^2+4x^2y^2}=\sqrt{1^2+1^2}=\sqrt{2}$

$\Rightarrow \quad x^2-y^2=1$ and $x^2+y^2=\sqrt{2}$

$\Rightarrow \quad x^2=\frac{\sqrt{2}+1}{2}$ and $y^2=\frac{\sqrt{2}-1}{2}$

$\Rightarrow \quad x=\pm\sqrt{\frac{\sqrt{2}+1}{2}}$ and $y=\pm\sqrt{\frac{\sqrt{2}-1}{2}}$.

Since $xy>0$, so x and y have the same sign.

$\therefore \quad \left\{x-\sqrt{\frac{\sqrt{2}+1}{2}}, y-\sqrt{\frac{\sqrt{2}-1}{2}}\right\}$ or $\left\{x--\sqrt{\frac{\sqrt{2}+1}{2}}, y--\sqrt{\frac{\sqrt{2}-1}{2}}\right\}$.

$\therefore \quad \sqrt{1-i}=\left\{\sqrt{\frac{\sqrt{2}+1}{2}}-\sqrt{\frac{\sqrt{2}-1}{2}}i\right\}$ or $\left\{-\sqrt{\frac{\sqrt{2}+1}{2}}+i\sqrt{\frac{\sqrt{2}-1}{2}}\right\}$.

EXERCISE 5G

Very-Short-Answer Questions

1. Evaluate $\frac{1}{i^{78}}$.
2. Evaluate $(i^{57}+i^{70}+i^{91}+i^{101}+i^{104})$.
3. Evaluate $\left(\frac{i^{180}+i^{178}+i^{176}+i^{174}+i^{172}}{i^{170}+i^{168}+i^{166}+i^{164}+i^{162}}\right)$.
4. Evaluate $(i^{4n+1}-i^{4n-1})$.
5. Evaluate $(\sqrt{-36}\times\sqrt{-25})$.
6. Find the sum $(i^n+i^{n+1}+i^{n+2}+i^{n+3})$, where $n\in N$.
7. Find the sum $(i+i^2+i^3+i^4+\ldots \text{ up to } 400 \text{ terms})$.
8. Evaluate $(1+i^{10}+i^{20}+i^{30})$.
9. Evaluate $\left(i^{41}+\frac{1}{i^{71}}\right)$.
10. Find the least positive integer n for which $\left(\frac{1+i}{1-i}\right)^n = 1$.
11. Express $(2-3i)^3$ in the form $m(a+ib)$.
12. Express $\frac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}\,i)-(\sqrt{3}-\sqrt{2}\,i)}$ in the form $(a+ib)$.
13. Express $\left(\frac{3-\sqrt{-16}}{1-\sqrt{-9}}\right)$ in the form $(a+ib)$.
14. Solve for x: $(1-i)x+(1+i)y=1-3i$.
15. Solve for x: $x^2-5ix-6=0$.
16. Find the conjugate of $\frac{1}{(3+4i)}$.
17. If $z=(1-i)$, find z^{-1}.
18. If $z=(\sqrt{5}+3i)$, find z^{-1}.
19. Prove that $\arg(z)+\arg(\bar{z})=0$.
20. If $|z|=6$ and $\arg(z)=\frac{3\pi}{4}$, find z.
21. Find the principal argument of $(-2i)$.
22. Write the principal argument of $(1+i\sqrt{3})^2$.
23. Write -9 in polar form.
24. Write $2i$ in polar form.
25. Write $-3i$ in polar form.
26. Write $z=(1-i)$ in polar form.

27. Write $z = (-1 + i\sqrt{3})$ in polar form.

28. If $|z| = 2$ and $\arg(z) = \frac{\pi}{4}$, find z.

ANSWERS (EXERCISE 5G)

1. -1 **2.** $-i$ **3.** -1 **4.** $2i$ **5.** -30 **6.** 0 **7.** 0 **8.** 0 **9.** $2i$

10. $n = 4$ **11.** $46 - 9i$ **12.** $a = 0, b = \frac{-7}{\sqrt{2}}$ **13.** $\left(\frac{3}{2} + \frac{1}{2}i\right)$

14. $x = 2, y = -1$ **15.** $x = 3i$ or $x = 2i$ **16.** $\left(\frac{3}{25} + \frac{4}{25}i\right)$ **17.** $\left(\frac{1}{2} + \frac{1}{2}i\right)$

18. $\left(\frac{\sqrt{5}}{14} - \frac{3}{14}i\right)$ **20.** $z = 3\sqrt{2}(-1 + i)$ **21.** $\frac{-\pi}{2}$ **22.** $\frac{2\pi}{3}$

23. $9(\cos\pi + i\sin\pi)$ **24.** $2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

25. $3\left\{\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right\}$ **26.** $\sqrt{2}\left\{\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right\}$

27. $2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ **28.** $z = \sqrt{2}(1 + i)$

HINTS TO SOME SELECTED QUESTIONS

1. $\frac{1}{i^{78}} = \frac{1}{i^{78}} \times \frac{i^2}{i^2} = -1.$ $[\because\ i^{80} = (i^4)^{20} = 1]$

2. Given expression $= (i^{48} \times i^3 + i^{68} \times i^2 + i^{88} \times i^3 + i^{100} \times i + i^{104})$

$= (i^3 + i^2 + i^3 + i + 1) = (-i - 1 - i + i + 1) = -i.$

3. Given expression $= \frac{i^{172}(i^8 + i^6 + i^4 + i^2 + 1)}{i^{162}(i^8 + i^6 + i^4 + i^2 + 1)} = i^{(172-162)} = i^{10} = i^8 \times i^2 = -1.$

4. Given expression $= (i^{4n} \times i - i^{4n} \times i^{-1}) = \left(i - \frac{1}{i}\right) = \frac{(i^2 - 1)}{i} \times \frac{i}{i} = \frac{-2i}{-1} = 2i.$

5. $(\sqrt{-36} \times \sqrt{-25}) = (6i \times 5i) = 30i^2 = -30.$

6. Given sum $= i^n \times (1 + i + i^2 + i^3) = i^n(1 + i - 1 - i) = 0.$

7. Given sum $= (i + i^2 + i^3 + i^4) + (i^5 + i^6 + i^7 + i^8) + \ldots\ldots$

$= (i + i^2 + i^3 + 1) + i^5(1 + i + i^2 + i^3) + \ldots = 0.$ $[\because\ (1 + i + i^2 + i^3) = 0]$

8. Given sum $= (1 + i^2 + 1 + i^2) = 0.$

9. $\left(i^{41} + \frac{1}{i^{71}}\right) = \left(i + \frac{1}{i^3}\right) = \left(i + \frac{1}{i^3} \times \frac{i}{i}\right) = (i + i) = 2i.$

10. $\left(\frac{1+i}{1-i}\right) = \frac{(1+i)}{(1-i)} \times \frac{(1+i)}{(1+i)} = \frac{(1+i)^2}{(1-i^2)} = \frac{(1 + i^2 + 2i)}{2} = \frac{2i}{2} = i.$

$\therefore\quad \left(\frac{1+i}{1-i}\right)^n = 1 \Rightarrow i^n = 1 \Rightarrow i^n = i^4 \Rightarrow n = 4.$

11. $(2-3i)^3 = 2^3 - (3i)^3 - (3\times 2\times 3i)(2-3i)$

$= (8 - 27i^3 - 36i - 54) = (8 + 27i - 36i - 54) = (46 - 9i).$

12. Given expression $= \dfrac{(9-5i^2)}{2\sqrt{2}\,i} = \dfrac{14}{2\sqrt{2}\,i}\times\dfrac{i}{i} = \dfrac{-7i}{\sqrt{2}} \Rightarrow a = 0, b = \dfrac{-7}{\sqrt{2}}.$

13. Given expression $= \dfrac{3-4i}{1-3i}\times\dfrac{1+3i}{1+3i} = \dfrac{(3-4i)(1+3i)}{(1+9)} = \dfrac{15+5i}{10} = \dfrac{1}{2}(3+i).$

14. $(1-i)x + (1+i)y = 1 - 3i \Leftrightarrow (x+y) + (y-x)i = 1 - 3i$

$\therefore \quad x + y = 1$ and $y - x = -3 \Rightarrow x = 2, y = -1.$

15. $x^2 - 5ix - 6 = 0 \Rightarrow x^2 - 3ix - 2ix - 6 = 0 \Rightarrow x(x-3i) - 2i(x-3i) = 0$

$\Rightarrow (x-3i)(x-2i) = 0 \Rightarrow x = 3i$ or $x = 2i.$

16. $z = \dfrac{1}{(3+4i)}\times\dfrac{(3-4i)}{(3-4i)} = \dfrac{3-4i}{25} = \left(\dfrac{3}{25} - \dfrac{4}{25}i\right) \Rightarrow \bar{z} = \left(\dfrac{3}{25} + \dfrac{4}{25}i\right).$

17. $z^{-1} = \dfrac{\bar{z}}{|z|^2} = \dfrac{(1+i)}{(1^2+1^2)} = \left(\dfrac{1}{2} + \dfrac{1}{2}i\right).$

18. $z^{-1} = \dfrac{\bar{z}}{|z|^2} = \dfrac{(\sqrt{5}-3i)}{(5+9)} = \left(\dfrac{\sqrt{5}}{14} - \dfrac{3}{14}i\right).$

20. $z = 6\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right) = 6\left\{\cos\left(\pi - \dfrac{\pi}{4}\right) + i\sin\left(\pi - \dfrac{\pi}{4}\right)\right\}$

$\Rightarrow \quad z = 6\left(-\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right) = 6\left(\dfrac{-1}{\sqrt{2}} + i\cdot\dfrac{1}{\sqrt{2}}\right) = \dfrac{6}{\sqrt{2}}(-1+i)$

$\Rightarrow \quad z = 3\sqrt{2}\,(-1+i).$

22. $z = (1 + i\sqrt{3})^2 = (-2 + 2i\sqrt{3}).$

REVIEW OF FACTS AND FORMULAE

1. (i) **Imaginary Numbers** *A given number is said to be imaginary, if its square is negative*, e.g., $\sqrt{-1}, \sqrt{-2}, \sqrt{-3}, \sqrt{-4}$, etc.

(ii) We denote $\sqrt{-1}$ by the Greek letter iota 'ι' transliterated as 'i'.

(iii) $i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i$ and $i^4 = 1$.

(iv) Let n be a positive integer. On dividing n by 4, let m be the quotient and r be the remainder.

Then, $n = 4m + r$, where $0 \le r < 4$.

$\therefore \quad i^n = i^{4m+r} = (i^4)^m \cdot i^r = i^r. \qquad [\because\ i^4 = 1]$

(v) $\sqrt{-16}\times\sqrt{-9} = (4i)\times(3i) = 12i^2 = -12.$

2. Complex Numbers

(i) The numbers of the form $z = (a + ib)$ are called complex numbers, where $a, b \in R$. Here, $\mathrm{Re}(z) = a$ and $\mathrm{Im}(z) = b$.

If $b = 0$ then z is purely real.

If $a = 0$ then z is purely imaginary.

(ii) Let $z_1 = (a_1 + ib_1)$ and $z_2 = (a_2 + ib_2)$ then

I. $(z_1 + z_2) = (a_1 + a_2) + i(b_1 + b_2)$

II. $(z_1 - z_2) = (a_1 - a_2) + i(b_1 - b_2)$

III. $z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$.

(iii) The conjugate of $z = (a + ib)$ is $\bar{z} = \overline{(a + ib)} = (a - ib)$.

(iv) The modulus of $z = (a + ib)$ is $|z| = \sqrt{a^2 + b^2}$.

(v) If $z_1 = (a_1 + ib_1)$ and $z_2 = (a_2 + ib_2)$ then $z_1 = z_2 \Leftrightarrow a_1 = a_2$ and $b_1 = b_2$.

(vi) If $z = (a + ib)$ then $z^{-1} = \dfrac{\bar{z}}{|z|^2} = \dfrac{(a - ib)}{(a^2 + b^2)}$.

3. (i) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ (ii) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ (iii) $\overline{z_1 z_2} = \bar{z}_1\, \bar{z}_2$

(iv) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$ (v) $z + \bar{z} = 2\,\mathrm{Re}(z)$ and $z - \bar{z} = 2\,\mathrm{Im}(z)$

(vi) $z\bar{z} = |z|^2$

4. Polar form of $z = (x + iy)$ is $z = r(\cos\theta + i\sin\theta)$.

Here, $r = \sqrt{x^2 + y^2} = |z|$ and the value of θ such that $-\pi < \theta \le \pi$ is called the principal argument or amplitude of z.

5. *When z is purely real;* then, it lies on the x-axis.

(i) If z lies on positive side of the x-axis then $\theta = 0$.

(ii) If z lies on negative side of the x-axis then $\theta = \pi$.

6. *When z is purely imaginary*; then, it lies on the y-axis.

(i) If z lies on the y-axis and above the x-axis then $\theta = \dfrac{\pi}{2}$.

(ii) If z lies on the y-axis and below the x-axis then $\theta = \dfrac{-\pi}{2}$.

7. When $z = (x + iy)$ does not lie on any axes:

Step 1. Find α for which $\tan\alpha = \left|\dfrac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right|$.

Step 2. Find the quadrant in which $P(x, y)$ lies.

(i) *When z lies in quadrant I;*
then, $\theta = \alpha \Rightarrow \arg(z) = \alpha$.

(ii) *When z lies in quadrant II;*
then, $\theta = \pi - \alpha \Rightarrow \arg(z) = \pi - \alpha$.

(iii) *When z lies in quadrant III;*
then, $\theta = (\alpha - \pi) \Rightarrow \arg(z) = \alpha - \pi$.

(iv) *When z lies in quadrant IV;*
then, $\theta = -\alpha \Rightarrow \arg(z) = -\alpha$.

8. (i) *Fundamental theorem of algebra*

A polynomial equation of degree *n* has at most *n* roots.

(ii) Let $ax^2 + bx + c = 0$, where $a, b, c \in R$ and $a \neq 0$.

Its roots are given by

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Here, $D = (b^2 - 4ac)$ is called the *discriminant*.

If $D < 0$, the given quadratic equation has complex roots.

9. $\sqrt{a + ib} = \pm(x + iy)$, where $x^2 = \frac{1}{2}\left\{\sqrt{a^2 + b^2} + a\right\}$ and $y^2 = \frac{1}{2}\left\{\sqrt{a^2 + b^2} - a\right\}$.

Similarly, by assuming that $\sqrt{a - ib} = \pm(x - iy)$, we may find $\sqrt{a - ib}$.